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Elastic Scattering and Polarization of High-Energy Nucleons by Nuclei (*).

A. M. SAPERSTEIN (**) and D. FELDMAN

Department of Physics, Brown University - Providence, R.I.

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Summary. — The cross section and polarization for the elastic scattering of protons by carbon and lead at 135 MeV and by carbon at 300 MeV have been evaluated in the impulse approximation using the nucleon-nucleon phase-shift sets of Signell-Marshak, Gammel-Thaler, Ohnuma-Feldman, and Stapp. The detailed calculations have been carried out both in ordinary Born approximation and also in terms of a modified Born approximation in which account is taken of the absorption of the elastically scattered nucleon wave within the nucleus. Relativistic effects have been estimated and have been found to be small. The possibility of distinguishing among the various sets of nucleon-nucleon phase shifts is discussed.

1. – Introduction.

The use of a complex potential containing both central and spin-orbit terms—the « optical model »—has proved to be very helpful in understanding the data on the elastic scattering of particles (nucleons and mesons) by nuclei (1). With the development of the impulse approximation (2), it has become pos-

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(**) Now at the Department of Physics, University of Buffalo, Buffalo, N.Y.

(1) For a comprehensive survey and bibliography, see S. J. LINDENBAUM: *Ann. Rev. Nucl. Sci.*, **7**, 317 (1957); H. A. BETHE: *Ann. Phys.*, **3**, 190 (1958); A. E. TAYLOR: *Rep. Progr. Phys.*, **20**, 86 (1957).

(2) G. F. CHEW and M. L. GOLDBERGER: *Phys. Rev.*, **87**, 778 (1952); N. C. FRANCIS and K. M. WATSON: *Phys. Rev.*, **92**, 291 (1953).

sible to express the optical-model potential in terms of averages over two-body transition matrices, the latter being given near the energy shell (and hence for not too large scattering angles) by the phase shifts for the two-body elastic scattering. When the incident particle is a proton or a neutron, identity effects come into play, but it has been shown—to the same accuracy as in the derivation of the impulse approximation—that the only change necessary is the use of properly antisymmetrized nucleon-nucleon scattering amplitudes ([3]).

Given several alternative sets of nucleon-nucleon phase shifts, several optical-model potentials may be derived, and one might hope that some of the phase-shift sets would be ruled out by a comparison of the nucleon-nucleus scattering with that predicted by the several, different potentials. This procedure was carried out by BETHE ([1]) who computed potentials for proton-carbon scattering at 300 MeV using six different sets of proton-proton phase shifts obtained by STAPP and complementary proton-neutron phase shifts derived by GAMMEL and THALER. Using these potentials, he calculated cross-sections and polarizations at very small angles ($< 7°$), but could not distinguish among the different phase-shift sets from a consideration of the goodness of fit obtained to the proton-carbon data of CHAMBERLAIN *et al.* ([4]). BETHE computed nucleon-nuclei potentials using the impulse approximation for uncharged nucleons; to these he added the Coulomb potential of a charge distribution having the same shape and size as the nucleon distribution within the nucleus. The resultant potentials were then used to determine the phase shifts by a modified JWKB method; the partial waves were summed analytically to find the cross-section, while the polarization was obtained in first Born approximation.

The Born approximation was also used by OHNUMA ([5]) who calculated the polarization and triple-scattering parameter β at various energies, using sets of phase shifts given by GAMMEL-THALER ([6]), FESHBACH-LOMON ([7]) and SIGNELL-MARSHAK ([8]). In the calculation, the Coulomb phase shifts were omitted from the nuclear amplitudes, and this omission was compensated for by neglecting the phase factor $\exp[-2i\eta \ln \sin \frac{1}{2}\theta]$ which normally multiplies the Coulomb amplitude in the case of pure Coulomb scattering. Ohnuma's calculation could not distinguish between the G-T and S-M phase-shift sets though he

([3]) G. TAKEDA and K. M. WATSON: *Phys. Rev.*, **97**, 1339 (1955); F. COESTER and H. KUMMEL: *Nucl. Phys.*, **9**, 225 (1958-59).

([4]) O. CHAMBERLAIN, E. SEGRÈ, R. D. TRIPP, C. WIEGAND and T. YPSILANTIS: *Phys. Rev.*, **102**, 1659 (1956).

([5]) S. OHNUMA: *Phys. Rev.*, **111**, 1173 (1958).

([6]) J. GAMMEL and R. THALER: *Phys. Rev.*, **107**, 291 (1957), referred to as G-T.

([7]) H. FESHBACH and E. LOMON: *Phys. Rev.*, **102**, 891 (1956), referred to as F-L.

([8]) P. S. SIGNELL and R. E. MARSHAK: *Phys. Rev.*, **109**, 1229 (1958), referred to as S-M.

could definitely rule out the F-L sets which had previously been rejected from a consideration of p-p polarization ([9]).

The impulse approximation gives the nucleon-nucleus potential in a momentum-space representation; this, of course, is also the nucleon-nucleus scattering amplitude in first Born approximation. It can be shown ([10]) (see also BETHE ([1])) that the polarization is given correctly in the first Born approximation for very small scattering angles which means that the small-angle polarization can be written down immediately, given the impulse-approximation amplitudes. For the cross-section and the polarization at larger angles, more exact methods must be used; thus, phase shifts must be found from the resulting potential, and then partial amplitudes need to be summed to give the correct nucleon-nucleus scattering amplitudes. The exact method is tedious, especially when many phase-shift sets are to be examined; it would be useful to have available some intermediate form of approximation which would facilitate the calculation of the nucleon-nucleus cross-section and the polarization at all angles, and, in this paper, we discuss such an approximation. In Section **2** formulae are given for the Born-approximation scattering amplitudes, and in Section **3** an improvement on the Born amplitudes is given which makes use of the fact that the Born amplitudes appear as products of two functions of the momentum transfer, the one being a nuclear form factor, the other a mean nucleon-nucleon scattering amplitude. The method has been applied to calculate the cross-section and polarization for the elastic scattering of protons by carbon and lead at 135 MeV and by carbon at 300 MeV. The results are discussed in Sections **3** and **4**.

2. – Impulse approximation: first Born calculation.

According to the impulse approximation ([2]), the potential for elastic nucleon-nucleus scattering in a momentum-space representation is given by

$$\langle \boldsymbol{P}_f | V | \boldsymbol{P}_0 \rangle = A \langle \psi_0, \boldsymbol{P}_f | \boldsymbol{t} | \psi_0, \boldsymbol{P}_0 \rangle , \tag{1}$$

where $\boldsymbol{P}_f$, $\boldsymbol{P}_0$ are the final and initial nucleon momenta, ψ_0 is the nuclear ground-state wave function, $\boldsymbol{t}$ is the two-nucleon transition operator averaged over the neutrons and protons in the nucleus (we are assuming charge-independent scattering), and A is the number of nucleons in the nucleus; we are here neg-

([9]) A. M. SAPERSTEIN and L. DURAND (III): *Phys. Rev.*, **104**, 1103 (1956).

([10]) H. S. KÖHLER: *Nucl. Phys.*, **1**, 433 (1956); I. I. LEVINTOV: *Soviet Physics JEPT*, **1**, 175 (1956).

lecting the possibility of inelastic excitation in intermediate states. To simplify the notation, we do not indicate the spin and isotopic-spin indices explicitly.

If we assume that the nucleus is initially at rest and has no recoil in the scattering, we find that

$$\langle \boldsymbol{P}_f | V | \boldsymbol{P}_0 \rangle = A \sum_s \langle \boldsymbol{P}_f, \boldsymbol{P}_0 - \boldsymbol{P}_f | \boldsymbol{t}_s | \boldsymbol{P}_0, 0 \rangle D_s , \tag{2}$$

where the sum contains two terms which refer to scattering from nucleons in the nucleus with spin up and spin down, and $\boldsymbol{t}_s$ is that part of the two-body operator which omits the spin-flip terms for the target nucleons. We assume that S_z, the z component of the nuclear spin $\boldsymbol{S}$, does not change during the scattering; the point here is that we are neglecting correlations between nucleons in the nucleus during the scattering process so that there can be no change in the z components of the spin of the individual nucleons in the nucleus. The function D_s is the nuclear form factor

$$D_s = \int \mathrm{d}\boldsymbol{r}\, \varrho_s(\boldsymbol{r}) \exp[i(\boldsymbol{P}_f - \boldsymbol{P}_0)\cdot \boldsymbol{r}/\hbar] , \tag{3}$$

where $\varrho_s(\boldsymbol{r})$ is the nucleon density for the particular spin involved, normalized so that

$$\sum_s \int \varrho_s(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} = 1 .$$

If $\varrho_s(\boldsymbol{r})$ is spherically symmetric, then

$$D_s \equiv D_s(\Lambda) = \frac{4\pi}{\Lambda} \int_0^\infty \varrho_s(r) \sin \Lambda r\, r\, \mathrm{d}r , \tag{4}$$

where

$$\hbar\Lambda \equiv |\boldsymbol{P}_f - \boldsymbol{P}_0| = \sqrt{2}\, P_0 A (A+1)^{-1} \left\{ 1 + A^{-1} [\sin^2\theta - \cos\theta (A^2 - \sin^2\theta)^{\frac{1}{2}}] \right\}^{\frac{1}{2}} , \tag{5}$$

and θ is the scattering angle in the laboratory system.

We assume that angular momentum and spin are quantized along the direction of incidence ($\boldsymbol{P}_0$ forms the z axis). If $\varrho_+(r)$ is the density of nucleons in the nucleus with spin up, *i.e.*, in the positive z direction, $\varrho_-(r)$ the density with spin down, and if we assume $\varrho_-(r)/\varrho_+(r)$ is independent of r, then

$$\varrho_-(r)/\varrho_+(r) = (A - 2S_z)/(A + 2S_z) \equiv N(A, S_z) . \tag{6}$$

We can therefore write, now noting explicitly the initial and final spins of the scattering nucleon,

$$(7)\quad \begin{cases} \langle \boldsymbol{P}_f, +|V|\boldsymbol{P}_0, +\rangle = A(t_{11} + \tfrac{1}{2}N(A, S_z)[t_0 + t_{00}])D_s\,, \\ \langle \boldsymbol{P}_f, -|V|\boldsymbol{P}_0, +\rangle = (2)^{-\frac{1}{2}}A(t_{01} + N(A, S_z)t_{-10})D_s\,, \\ \langle \boldsymbol{P}_f, +|V|\boldsymbol{P}_0, -\rangle = (2)^{-\frac{1}{2}}A(t_{10} + N(A, S_z)t_{0\,-1})D_s\,, \\ \langle \boldsymbol{P}_f, -|V|\boldsymbol{P}_0, -\rangle = A(\tfrac{1}{2}[t_0 + t_{00}] + N(A, S_z)t_{-1-1})D_s\,, \end{cases}$$

where t_0 is the singlet and t_{ij} $(i, j = -1, 0, 1)$ are the triplet parts of the two-body transition operator.

To use the above results, we have to average over the $(2S+1)$ possible values of S_z in $N(A, S_z)$. This is a very tedious process, since the averaging must be done after the scattering amplitudes are computed and squared. However, we can write

$$(8)\qquad N(A, S_z) \approx 1 - 4S_z/A + 4S_z^2/A^2 + \dots\,,$$

so that for small nuclear spin and not too small A, we can simply set $N(A, S_z) \approx 1$ and eliminate the need for averaging over S_z. We see that the spin of the nucleus enters as S/A, so that the elastic scattering of nucleons by nuclei is not expected to be sensitive to the value of the nuclear spin—most nuclei should behave like spin-zero nuclei; this is found to be true in practice.

We now assume that the two-nucleon transition operator $\boldsymbol{t}$ which is effective for nucleon-nucleus scattering at the laboratory angle θ can be represented approximately by the $\boldsymbol{t}$ for nucleon-nucleon scattering at the same laboratory angle, *i.e.*, we neglect the difference in energy transfer between the two nucleons in the two situations; this should be a good approximation for small-angle scattering. The matrix elements of $\boldsymbol{t}$ can then be expressed in terms of nucleon-nucleon phase shifts in the usual way; these matrix elements are given as functions of the center-of-mass scattering angle $\Theta\,(=\frac{1}{2}\theta)$ and wave number $k\,(=\frac{1}{2}k_{\text{lab}})$ by BREIT, EHRMAN and HULL [11]. We can then write the optical potential as a matrix in the spin space of the scattering nucleon, *viz.*,

$$(9)\quad \langle \boldsymbol{P}_f|V|\boldsymbol{P}_0\rangle = -\frac{\hbar^2}{(2\pi)^2 m\, k_{\text{lab}}} \begin{pmatrix} G(\Lambda) & H(\Lambda)\exp[i\varphi] \\ -H(\Lambda)\exp[-i\varphi] & G(\Lambda) \end{pmatrix} D(\Lambda)\,,$$

where

$$(10)\qquad D(\Lambda) = (4\pi/\Lambda)\int_0^\infty \varrho(r)\sin \Lambda r\, r\,\mathrm{d}r\,,$$

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[11] G. BREIT, J. EHRMAN and M. H. HULL Jr.: *Phys. Rev.*, **97**, 1051 (1955).

and where $D(\Lambda = 0)$ is now normalized to unity, *i.e.*,

$$\int \varrho(r)\,\mathrm{d}r = 1\,. \tag{11}$$

The direction of scattering is characterized by the angles θ and φ; for a fixed energy, θ and Λ are uniquely related by eq. (5). Except where the contrary is noted, we use non-relativistic kinematics.

The averaging over the neutrons and protons in the nucleus is done by writing, for incident protons, $A\boldsymbol{t} = (A-Z)\boldsymbol{t}^{\mathrm{np}} + Z\boldsymbol{t}^{\mathrm{pp}} = \frac{1}{2}(A-Z)\boldsymbol{t}^0 + \frac{1}{2}(A+Z)\boldsymbol{t}^1$, where the superscript denotes the isotopic spin. The transition-matrix elements can then be expressed in terms of the α scattering amplitudes of BREIT, EHRMAN and HULL ([11]); the α_j $(j = 1, \ldots, 5)$ are the triplet scattering amplitudes, α_s is the singlet amplitude. We write $\alpha = \alpha^0 + \alpha^1$, where α^0, the isotopic-singlet amplitude, includes odd orbital angular-momentum (odd-L) phase shifts for singlet spin states, even-L phase shifts for triplet spin states; α^1, the isotopic-triplet amplitude, includes even-L phase shifts for singlet spin states and odd-L phase shifts for triplet spin states. For incident protons, we can then write

$$\begin{cases} G(\Lambda) = (A-Z)(2\alpha_2^0 + \alpha_5^0 + \alpha_s^0) + (A+Z)(2\alpha_2^1 + \alpha_5^1 + \alpha_s^1)\,, \\ H(\Lambda) = \sin\Theta[(A-Z)(\alpha_1^0 - \alpha_4^0) + (A+Z)(\alpha_1^1 - \alpha_4^1)]\,. \end{cases} \tag{12}$$

Correspondingly, when the incident nucleons are neutrons, we have

$$\begin{cases} G(\Lambda) = Z(2\alpha_2^0 + \alpha_5^0 + \alpha_s^0) + (2A-Z)(2\alpha_2^1 + \alpha_5^1 + \alpha_s^1)\,, \\ H(\Lambda) = \sin\Theta[Z(\alpha_1^0 - \alpha_4^0) + (2A-Z)(\alpha_1^1 - \alpha_4^1)]\,. \end{cases} \tag{13}$$

The potentials, which are given by (9), (10), (12), and (13), include the effects of identical particles but do not as yet take into account Coulomb effects. These can be added at the end, as Bethe does, by appending to the nuclear potentials a Coulomb potential derived from an extended charge source. Or, as we prefer to do it, we can include Coulomb effects in the elementary two-body scatterings. This is done in the manner of BREIT ([12]); we add Coulomb phase shifts to those parts of the amplitudes which refer to proton-proton scattering, and add the proper Coulomb amplitudes to α_2, α_5, and α_s. Defining the Coulomb amplitude

$$S(\Theta) \equiv \tfrac{1}{2}\eta\{-(4/s^2)\exp[-i\eta\ln s^2] + (2/c^2)\exp[-i\eta\ln c^2]\}\,, \tag{14}$$

([12]) G. BREIT and M. H. HULL Jr.: *Phys. Rev.*, **97**, 1047 (1955).

and the Coulomb phase shifts

$$\sigma_{L,0} \equiv \mathrm{tg}^{-1}(\eta/L) + \mathrm{tg}^{-1}(\eta/(L-1)) + \ldots + \mathrm{tg}^{-1}\eta\,, \tag{15}$$

where $s \equiv \sin\frac{1}{2}\Theta$, $c \equiv \cos\frac{1}{2}\Theta$, $\eta \equiv e^2/\hbar v$, e is the charge on the proton, and v is the relative velocity of the two particles, we can write the two-nucleon part of the potential (9) for an elastically scattered proton as follows:

$$\begin{aligned} G(\Lambda) = {} & P_0(\Theta)[(A-Z)3Q_{S,m}^{J-1} + (A+Z)Q_S] + \\ & + P_1(\Theta)[(A-Z)3Q_P + (A-Z+2e_{10}Z)(5Q_{P,m}^{J-2} + 3Q_{P,1} + Q_{P,0})] + \\ & + P_2(\Theta)[(A-Z)(7Q_{D,m}^{J-3} + 5Q_{D,2} + 3Q_{D,m}^{J-1}) + (A-Z+2e_{20}Z)5Q_D] + \\ & + P_3(\Theta)[(A-Z)7Q_F + (A-Z+2e_{30}Z)(9Q_{F,m}^{J-4} + 7Q_{F,3} + 5Q_{F,m}^{J-2})] + \\ & + P_4(\Theta)[(A-Z)(11Q_{G,m}^{J-5} + 9Q_{G,4} + 7Q_{G,m}^{J-3}) + (A-Z+2e_{40}Z)9Q_G] + \\ & + P_5(\Theta)[(A-Z)11Q_H + (A-Z+2e_{50}Z)(13Q_{H,m}^{J-6} + 11Q_{H,5} + 9Q_{H,m}^{J-4})] + \\ & + ZS(\Theta)\,, \end{aligned} \tag{16}$$

$$\begin{aligned} H(\Lambda) = {} & [\mathrm{d}P_1(\Theta)/\mathrm{d}\Theta][(5/2)Q_{P,m}^{J-2} - (3/2)Q_{P,1} - Q_{P,0}](A-Z+2e_{10}Z) + \\ & + [\mathrm{d}P_2(\Theta)/\mathrm{d}(\Theta)][(7/3)Q_{D,m}^{J-3} - (5/6)Q_{D,2} - (3/2)Q_{D,m}^{J-1}](A-Z) + \\ & + [\mathrm{d}P_3(\Theta)/\mathrm{d}(\Theta)][(9/4)Q_{F,m}^{J-4} - (7/12)Q_{F,3} - (5/3)Q_{F,m}^{J-2}](A-Z+2e_{30}A) + \\ & + [\mathrm{d}P_4(\Theta)/\mathrm{d}(\Theta)][(11/5)Q_{G,m}^{J-5} - (9/20)Q_{G,4} - (7/4)Q_{G,m}^{J-3}](A-Z) + \\ & + [\mathrm{d}P_5(\Theta)/\mathrm{d}(\Theta)][(13/6)Q_{H,m}^{J-6} - (11/30)Q_{H,5} - (9/5)Q_{H,m}^{J-4}](A-Z+2e_{50}Z)\,. \end{aligned} \tag{17}$$

Here, $e_{L0} \equiv \exp[2i\sigma_{L0}]$ are Coulomb phase factors, $Q_L \equiv (1/2i)(\exp[2i\delta_L] - 1)$ involves the singlet phase shifts δ_L, $Q_{L,J} \equiv (1/2i)(\exp[2i\delta_{LJ}] - 1)$ the uncoupled triplet phase shifts δ_{LJ}, and

$$Q_{L,m}^{J-L\mp1} \equiv (1/2i)(\cos^2\varepsilon_J \exp[2i\delta_J^{L-J\pm1}] + \sin^2\varepsilon_J \exp[2i\delta_J^{L-J\mp1}] - 1)$$

contains the two phase shifts characterized by J, $L = J \pm 1$ which are coupled by the parameter ε_J; $P_L(\Theta)$ are the Legendre polynomials. Notice that the off-diagonal elements of the scattering matrix, expressed in terms of an LJ representation, disappear in the process of averaging over the spin-up and spin-down nucleons in the nucleus.

Using eqs. (9) and (10), expressions for the scattering cross-section and

polarization can be written down in the first Born approximation. We obtain

$$\sigma_B(\theta) = k_{\text{lab}}^{-2} D^2(\Delta)(|G(\Delta)|^2 + |H(\Delta)|^2)\,, \tag{18}$$

$$P_B(\theta) = \frac{2\,\text{Im}\,G^*H}{|G|^2 + |H|^2}\,. \tag{19}$$

We see that the polarization depends only upon the two-nucleon phase shifts and not upon the structure of the particular nucleus involved.

Calculations of the cross-section and polarization have been carried out, in first Born approximation, for the elastic scattering of protons by carbon ($A = 12$, $Z = 6$). For the nucleon density $\varrho(r)$, a trapezoidal distribution was used with constants chosen to fit the electron-scattering data ([13]). In particular, we set

$$\varrho(r) = \begin{cases} (3/\pi)(C' + C)^{-1}(C'^2 + C^2)^{-1}\,, & 0 \leqslant r \leqslant C'\,, \\ (3/\pi)(C - r)/(C^4 - C'^4)\,, & C' \leqslant r \leqslant C\,, \\ 0\,, & r \geqslant C\,, \end{cases} \tag{20}$$

with $C' = 1.1$ fermis and $C = 3.5$ fermis. From this, we find

$$D(\Delta) = (12/\Delta^3)(C^4 - C'^4)^{-1} \cdot \\ \cdot\{C' \sin \Delta C' - C \sin \Delta C + (2/\Delta)[\cos \Delta C' - \cos \Delta C]\} \tag{21}$$

with $\Delta(\theta)$ given by Eq. (5). For computational purposes, it is convenient to use the following approximation valid for small Δ:

$$D(\Delta) \approx 1 - (\Delta^2/6)\int_0^\infty 4\pi r^4 \varrho(r)\,\mathrm{d}r = 1 - (6/15)(C^6 - C'^6)(C^4 - C'^4)^{-1}(\Delta^2/6)\,. \tag{22}$$

The computations were made at 135 MeV using four sets of phase shifts: one of SIGNELL and MARSHAK ([8]), one due to GAMMEL and THALER ([8,6]), and

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([13]) R. HOFSTADTER: *Rev. Mod. Phys.*, **28**, 214 (1956). It has been suggested by R. WILSON (to be published) that the mean-square nuclear radius, as determined from electron scattering, be decreased by the mean-square radius of the charge distribution of the nucleon, resulting in smaller values of C and C' than are used in this paper. It is not unambiguously clear, however, that this is a necessary procedure. The uncorrected charge distribution seems to predict the correct position for the cross section minimum in lead for which nucleus the minimum is clearly observed. Furthermore, calculations by KERMAN, MCMANUS and THALER ([23]), using the Born approximation for Helium, seem to show very little difference between the two approaches.

sets E and L of OHNUMA and FELDMAN [14]; also at 300 MeV, using the p-p phase shifts of Stapp's fit number one [15] and the corresponding n-p phase shifts of GAMMEL and THALER [6,8]. In addition, we have also considered the scattering at 135 MeV of protons by lead ($A = 208$, $Z = 82$) using the S-M phase shifts; here we assumed [13] $C' = 5.3$ fermis and $C = 7.7$ fermis. The resulting carbon polarizations are plotted in Figs. 1-3, the cross-sections are given in Figs. 4-6. For lead, the polarization is given in Fig. 7 and the cross-section in Fig. 8.

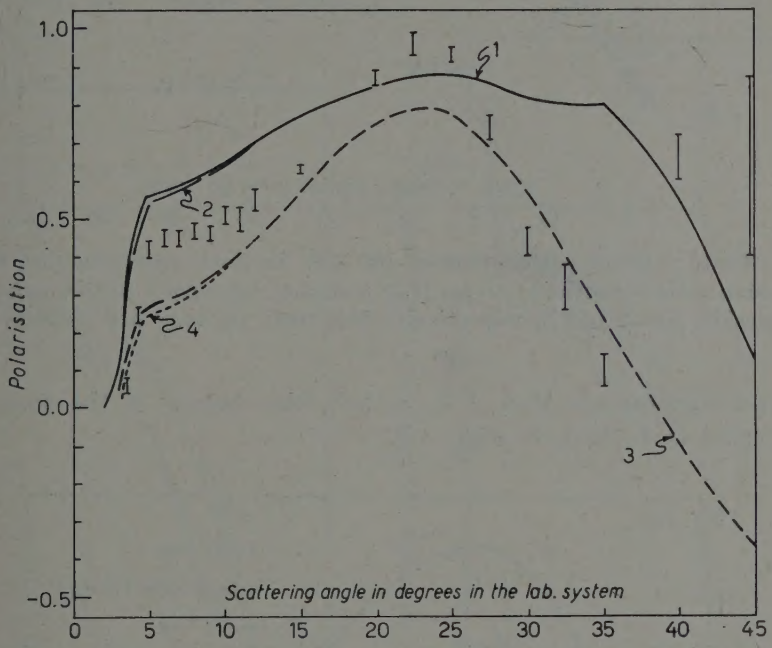


Fig. 1. – Proton-carbon polarization at 135 MeV in Born approximation with and without relativistic corrections; curves (3.4) and (1.2) correspond to Gammel-Thaler Signell-Marshak phase-shift and sets, respectively; data from DICKSON and SALTER [20].

In the case of the carbon calculations, an estimate was made of the effects of relativity by taking into account in the Coulomb amplitudes both relativistic kinematics as well as the magnetic dipole moments of the proton and neutron [16]. These effects make very little difference in the overall curves

[14] S. OHNUMA and D. FELDMAN: *Proc. Sixth Ann. Rochester Conference on High-Energy Physics* (New York, 1956), part II, p. 10, referred to as O-F.

[15] H. P. STAPP, T. J. YPSILANTIS and N. METROPOLIS: *Phys. Rev.*, **105**, 302 (1957), referred to as S. This one calculation was carried out at 300 MeV so as to serve as a check on the consistency of the procedures followed in this paper when applied at different energies.

[16] A. GARREN: *Phys. Rev.*, **101**, 419 (1956); G. BREIT: *Phys. Rev.*, **99**, 1581 (1955).

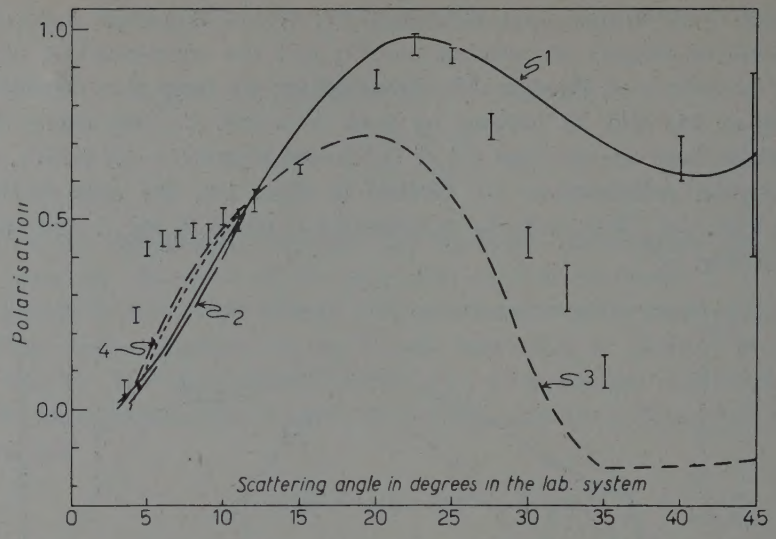


Fig. 2. – Proton-carbon polarization at 135 MeV in Born approximation with and without relativistic corrections; curves (1.2) and (3.4) correspond to Ohnuma-Feldman phase-shift sets E and L, respectively; data from DICKSON and SALTER ([20]).

for the polarization (see Figs. 1-3); in fact, the changes in the cross-section are too small to be seen in Figs. 4-6.

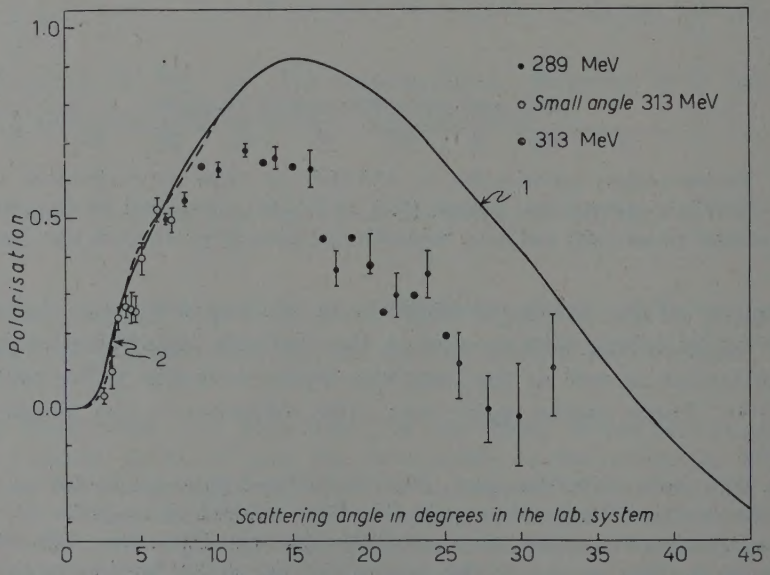


Fig. 3. – Proton-carbon polarization at 300 MeV in Born approximation with and without relativistic corrections; curves (1.2) correspond to Stapp phase-shift set no. 1; data from CHAMBERLAIN *et al.* ([4]).

At very small angles where the polarization in Born approximation is expected to be close to the results of an exact calculation ([10]), it is seen that sets E and L of O-F predict a polarization which is too small and which does not increase rapidly enough with increasing angle. The polarizations given by the G-T and S-M sets seem to have the right shape in carbon, though, in the latter case, it appears to be somewhat too large. At 300 MeV, the Stapp phases give a good representation of the small-angle data, though again a bit on the high side. The polarization for lead computed in Born approximation seems to bear no relation to the data (Fig. 7).

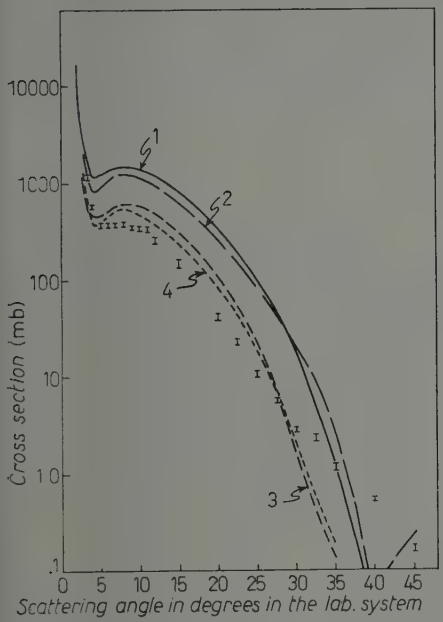


Fig. 4. – Proton-carbon differential scattering cross section at 135 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (4.2) and (3.1) correspond to Gammel-Thaler and Signell-Marshak phase-shift sets, respectively; data from Dickson and Salter ([20]).

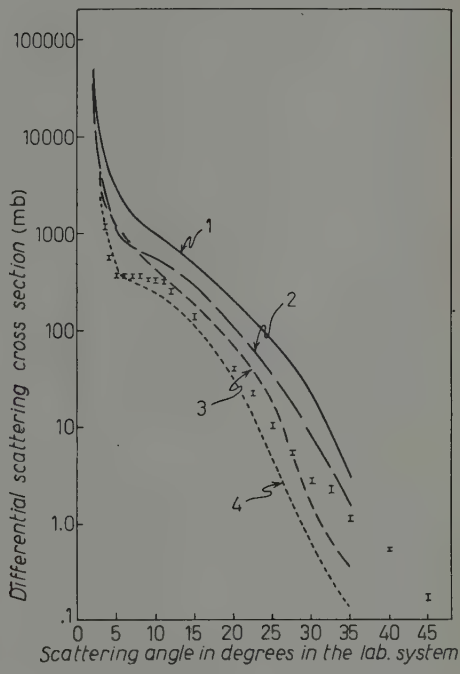
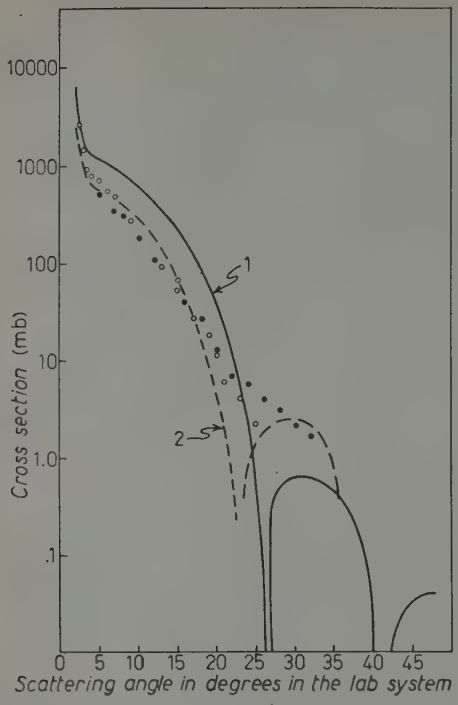


Fig. 5. – Proton-carbon differential scattering cross section at 135 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (3.1) and (4.2) correspond to Ohnuma-Feldman phase-shift sets E and L respectively; data from Dickson and Salter ([20]).

The cross-sections computed in Born approximation do not fit the data for any of the phase-shift sets. However, those predicted by S-M and G-T at 135 MeV and by Stapp at 300 MeV seem to be approximately constant multiples of the empirical cross-sections for $\theta < 25°$ (see Figs. 4, 5, 6, 8).



All computed cross-sections show diffraction minima: for carbon, there is, at 135 MeV, a minimum between 35° and 40° which is not observed; at 300 MeV, a minimum between 20° and 25° for which there is some evidence; for lead, there is, at 135 MeV, a minimum at about 16° — here there is an observed minimum at 13° corresponding to an increase in the nuclear radius by a factor of 1.23.

Fig. 6. – Proton-carbon differential scattering cross section at 300 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (2.1) correspond to Stapp phase-shift set no. 1; data from CHAMBERLAIN *et al.* [4] (the solid circles refer to 289 MeV, the open circles to 313 MeV).

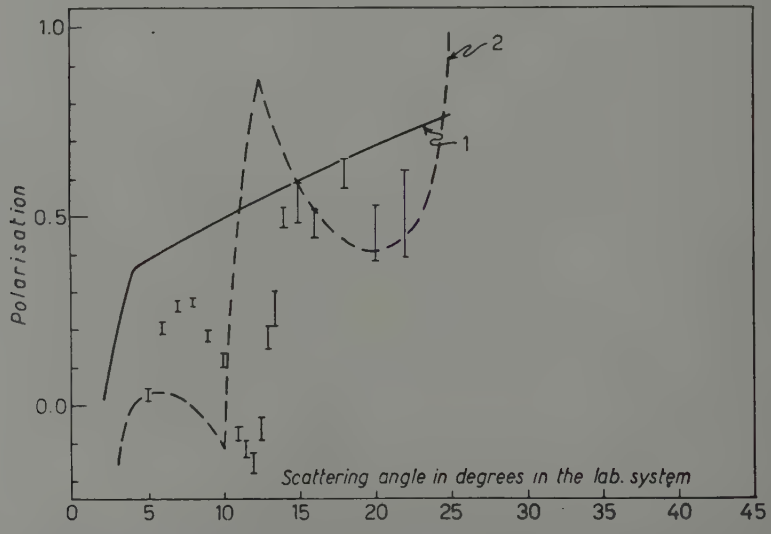


Fig. 7. – Proton-lead polarization at 135 MeV in Born approximation, with and without absorption; curves (2.1) correspond to Signell-Marshak phase-shift set; data from DICKSON [21].

The fact that the first Born approximation, which describes the scattering by a single nucleon in the nucleus, predicts a cross-section which is approximately a constant multiple of the observed cross-section suggests that the multiple scattering of the incident nucleon in the nucleus can be represented by an approximately angle-independent absorption of the incident wave, *i.e.*, in a higher-order Born approximation, all the interactions but one are expected to be predominantly absorptive.

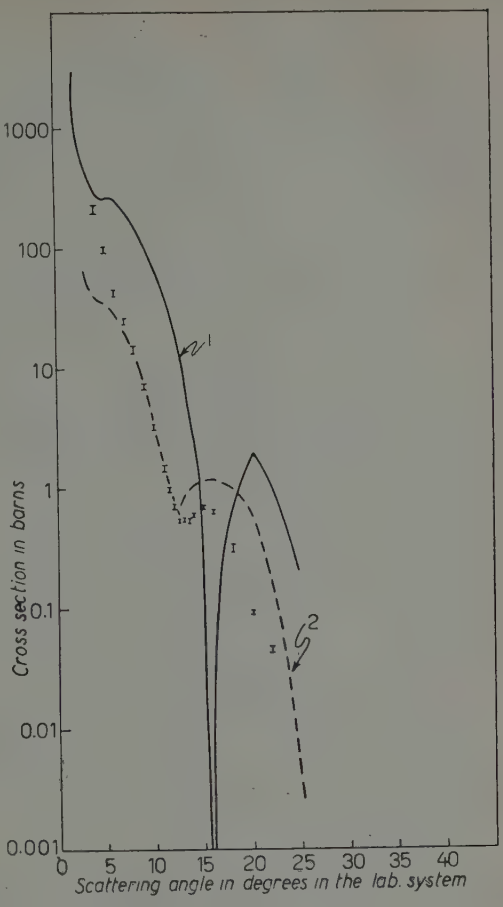


Fig. 8. – Proton-lead differential scattering cross section at 135 MeV in Born approximation, with and without absorption; curves (2.1) correspond to Signell-Marshak phase-shift set; data from DICKSON ([21]).

3. – Modification of Born approximation.

Given the known shortcomings of the first Born approximation, it is desirable to find a more accurate method of calculating the scattering due to the impulse-approximation potential. One can, of course, use numerical methods to obtain an exact solution; however, the overall procedure would be long and tedious, especially when many sets of nucleon-nucleon phase shifts are to be tested. Furthermore, an exact solution requires a knowledge of the impulse-approximation potential for all momentum transfers, whereas we are limited to small momentum transfers; we cannot depart too far from the energy shell, for otherwise the two-nucleon scattering phase shifts cannot be used. Fortunately, the structure of the impulse-approximation potential which appears as a product of a nuclear form factor and a mean nucleon-nucleon scat-

tering amplitude allows a simple modification of the Born approximation which may be expected to extend its range of applicability.

Let us assume, for the moment, that we are dealing with a central potential V. If the Born expansion converges ([17]) (we shall specifically exclude multiple Coulomb scattering), we can write the scattering amplitude as follows:

$$(23)\qquad f(\theta) = -\frac{(2\pi)^2 m}{\hbar^2}\sum_{n=1}^{\infty}\frac{1}{\hbar^{3(n-1)}}\int \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_{n-1}\langle \boldsymbol{P}_f|V|\boldsymbol{p}_{n-1}\rangle \frac{1}{a_{n-1}}\cdot$$
$$\cdot\langle \boldsymbol{p}_{n-1}|V|\boldsymbol{p}_{n-2}\rangle \frac{1}{a_{n-2}}\ldots\frac{1}{a_1}\langle \boldsymbol{p}_1|V|\boldsymbol{P}_0\rangle ,$$

where $a_i = (1/2m)(P_0^2 - p_i^2 + i\varepsilon)$ and where the $n=1$ term (first Born approximation) is simply $\langle \boldsymbol{P}_f|V|\boldsymbol{P}_0\rangle$. In the impulse approximation

$$\langle \boldsymbol{p}|V|\boldsymbol{q}\rangle = D'(|\boldsymbol{p}-\boldsymbol{q}|)\,G(|\boldsymbol{p}-\boldsymbol{q}|),$$

where

$$D'(|\boldsymbol{p}-\boldsymbol{q}|) = -[\hbar^2/(2\pi)^2 m k_{\mathrm{lab}}]\,D(|\boldsymbol{p}-\boldsymbol{q}|),$$

so that

$$(24)\quad f(\theta) = -\frac{(2\pi)^2 m}{\hbar^2}\sum_{n=1}^{\infty}\frac{1}{\hbar^{3(n-1)}}\int \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_{n-1} D'(|\boldsymbol{P}_f-\boldsymbol{p}_{n-1}|)\ldots D'(|\boldsymbol{p}_2-\boldsymbol{p}_1|)\cdot$$
$$\cdot D'(|\boldsymbol{p}_1-\boldsymbol{P}_0|)\,\frac{G(|\boldsymbol{P}_f-\boldsymbol{p}_{n-1}|)\ \ldots\ G(|\boldsymbol{p}_2-\boldsymbol{p}_1|)\,G(|\boldsymbol{p}_1-\boldsymbol{P}_0|)}{a_{n-1}\ldots a_1}.$$

Now the form factor $D(|\boldsymbol{p}-\boldsymbol{q}|)$ falls off very rapidly with increasing $|\boldsymbol{p}-\boldsymbol{q}|$ (see eqs. (21) and (22)). Furthermore, $G(|\boldsymbol{p}-\boldsymbol{q}|)$ does not vary very rapidly as a function of its argument for small scattering angles ($\Theta < 40°$ at 135 MeV, and $\Theta < 25°$ at 300 MeV) outside the pure Coulomb region. Therefore, the value of the integral will not change by much if $G(|\boldsymbol{p}_1-\boldsymbol{P}_0|)$ is replaced by $G(|\boldsymbol{P}_f-\boldsymbol{P}_0|)$, which can then be taken outside of the integral. One then has (for $\theta < 20°$ at 135 MeV, $\theta < 12.5°$ at 300 MeV)

$$(25)\qquad f(\theta) = \frac{G(A)}{k_{\mathrm{lab}}}\int \mathrm{d}\boldsymbol{p}_1\, F(\boldsymbol{P}_f, \boldsymbol{p}_1)\,D(|\boldsymbol{p}_1-\boldsymbol{P}_0|),$$

where

$$(26)\qquad F(\boldsymbol{P}_f, \boldsymbol{p}_1) = \sum_{n=1}^{\infty}\frac{1}{\hbar^{3(n-1)}}\int \mathrm{d}\boldsymbol{p}_2 \ldots \mathrm{d}\boldsymbol{p}_{n-1}\,(\boldsymbol{P}_f|V|\boldsymbol{p}_{n-1})\,\frac{1}{a_{n-1}}\ldots\frac{1}{a_2}\,(\boldsymbol{p}_2|V|\boldsymbol{p}_1)\,\frac{1}{a_1};$$

the first term in the summation is equal to $\delta(\boldsymbol{P}_f-\boldsymbol{p}_1)$.

([17]) R. Jost and A. Pais: *Phys. Rev.*, **82**, 840 (1951).

If $G(|\boldsymbol{p}-\boldsymbol{q}|)$ is set equal to unity, then

$$(27) \qquad (1/k_{\text{lab}})\, d(A) = (1/k_{\text{lab}}) \int d\boldsymbol{p}_1 F(\boldsymbol{P}_f, \boldsymbol{p}_1) D(|\boldsymbol{p}_1 - \boldsymbol{P}_0|)$$

would evidently be the complete solution for the scattering amplitude for a momentum-space potential $D'(|\boldsymbol{p}-\boldsymbol{q}|)$. Since G is not unity in our case, however, $(1/k_{\text{lab}})\, d(A)$ may be interpreted as a scattering amplitude whose first-order term involves only the potential D', but whose higher-order corrections stem from the complete potential $D'G$.

High-energy electron scattering by a Coulomb potential $V \propto D/|\boldsymbol{p}-\boldsymbol{q}|^2$ can be treated very successfully in the first Born approximation ([13]). The same approximation should hold at least equally well for scattering by the much less singular potential D', *i.e.*, if F were independent of G, we would expect $d(A) \approx D(A)$. Now, the nucleon-nucleon scattering amplitude G gives rise to a complex impulse-approximation potential. In view of this fact and also the discussion at the end of Section **2**, we assume that the main effect in (27) of the presence of G in F will be to decrease the amplitude of the nucleon wave due to absorption by the complex potential.

In other words, we consider that multiple scattering leads primarily to inelastic scattering, and hence require that F damp the nucleon wave in an angle-independent manner. Perhaps the simplest way to accomplish the latter is to assume that the absorption depends only upon the radial distance from the nuclear surface, *i.e.*, since

$$D(A) = \int d\boldsymbol{r}\, \varrho(r) \exp\left[i(\boldsymbol{P}_f - \boldsymbol{P}_0)\cdot \boldsymbol{r}\right],$$

we approximate (27) by setting

$$(28) \qquad d(A) \approx \int d\boldsymbol{r}\, \varrho(r) \exp\left[i(\boldsymbol{P}_f - \boldsymbol{P}_0)\cdot \boldsymbol{r} - (C-r)\varkappa(r)\right],$$

where $\varkappa(r)$ is the imaginary part of the complex wave number inside the nucleus, and C is the radius of the nucleus inside of which absorption occurs.

A similar consideration has been employed by PEVSNER and RAINWATER ([18]) in their analysis of the scattering of 79 MeV pions by aluminum. These authors assumed a complex square well and used exact phase-shift expansions as well as various forms of Born approximation. They found that a modification of the Born approximation in the above manner to account

([18]) A. PEVSNER and J. RAINWATER: *Phys. Rev.*, **100**, 1431 (1955).

for the attenuation of the elastically scattered wave inside the nucleus led to results almost identical with those obtained by phase-shift calculations.

As they stand, the above considerations apply only to a central-force potential, *i.e.*, to the case where the spin of the incident nucleon cannot be « flipped », and hence only one amplitude is necessary to describe the scattering of nucleons of a given spin sense. We can, none the less, treat our problem in a similar fashion as follows. First, we transform from the representation in which the nucleon spin is either parallel or antiparallel to the initial nucleon momentum to that in which the spin is perpendicular to the plane of the scattering. For a given spin direction, we may now distinguish between right-handed and left-handed scattering, respectively, and, for each case, we may introduce a central-force potential which is effective for scattering in the Born approximation. Thus, for right-handed scattering, we have

$$(\boldsymbol{P}_f|V|\boldsymbol{P}_0)_R = [G(\Delta) + iH(\Delta)]\,D'(\Delta)\,, \tag{29}$$

while, for left-handed scattering, we obtain

$$(\boldsymbol{P}_f|V|\boldsymbol{P}_0)_L = [G(\Delta) - iH(\Delta)]\,D'(\Delta)\,. \tag{30}$$

We now estimate the absorption of the elastically scattered wave inside the nucleus by computing for each of these two potentials the complex wave numbers $\varkappa_R(r)$ and $\varkappa_L(r)$, respectively, from which we may obtain, in turn, the corresponding modified form factors $d_R(\Delta)$ and $d_L(\Delta)$. Therefore, in the approximation within which we are working, we get

$$\begin{cases} f_R(\Delta) = (1/k_{\text{lab}})\,d_R(\Delta)[G(\Delta) + iH(\Delta)]\,, \\ f_L(\Delta) = (1/k_{\text{lab}})\,d_L(\Delta)[G(\Delta) - iH(\Delta)]\,, \end{cases} \tag{31}$$

so that, for the polarization, we have

$$P \equiv [\sigma(L) - \sigma(R)]/[\sigma(L) + \sigma(R)] = (P_B + \mathscr{D})/(1 + P_B\mathscr{D})\,, \tag{32}$$

where P_B is the Born-approximation polarization given by eq. (19), and

$$\mathscr{D} \equiv (d_L^2 - d_R^2)/(d_L^2 + d_R^2)\,. \tag{33}$$

For the cross-section, we find

$$\sigma = \frac{(d_R^2 + d_L^2)}{2k_{\text{lab}}^2}\{|G|^2 + |H|^2 - P_B\mathscr{D}\}\,. \tag{34}$$

Given a non-relativistic Schrödinger equation with a complex potential $V(r) = V_1(r) + iV_2(r)$, the complex wave number $k+i\varkappa$ is determined by the relation

$$(k + i\varkappa)^2 = (2m/\hbar^2)(E - V_1 - iV_2) , \tag{35}$$

where E is the total energy of the system. Then

$$\begin{cases} k^2(r) = \frac{1}{2}k_{\text{lab}}^2[1 - (V_1/E)]\{1 + [1 - V_2^2(E - V_1)^{-2}]^{\frac{1}{2}}\} , \\ \varkappa^2(r) = k_{\text{lab}}^2[1 - (V_1/E)]\{1 - [1 - V_2^2(E - V_1)^{-2}]^{\frac{1}{2}}\} , \end{cases} \tag{36}$$

where $\hbar^2 k_{\text{lab}}^2 = 2mE$; for k positive, the sign of $\varkappa$ is opposite to that of V_2. In view of the relation between the imaginary part of the forward scattering amplitude and the total nucleon-nucleon cross-section—« the optical theorem »—, V_2 is always negative, so that $\varkappa$ is then positive and leads to absorption.

To find $\varkappa$, we proceed as follows. Consider first the contribution to the potential from the term $G(\Lambda)D'(\Lambda)$ (see eqs. (29) and (30)); this has the form

$$V(r) = -(2\pi\hbar^2/mk_{\text{lab}})\varrho(r)\,\mathcal{G}(r) , \tag{37}$$

where

$$\varrho(r)\,\mathcal{G}(r) \equiv (2\pi^2 r)^{-1}\int_0^\infty G(\Lambda)\,D(\Lambda)\Lambda \sin \Lambda r \,\mathrm{d}\Lambda . \tag{38}$$

For small Λ (where Coulomb interference is important), the integrand in (38) is small because of the factor $\Lambda \sin \Lambda r$; for large Λ, the integrand is small because of $D(\Lambda)$. For values of Λ just outside the Coulomb region, $G(\Lambda)$ is roughly independent of angle for all of the phase-shift sets used. Therefore, in evaluating the integral, we may write, approximately, $\mathcal{G}(r) \approx G(\bar{\Lambda})$, where $\bar{\Lambda}$ is taken just outside the Coulomb region. In effect, what we are doing is using the variation of $G(\Lambda)$ with Λ to determine the shape of the cross-sections, and the approximately constant magnitude of $G(\Lambda)$ in the vicinity of $\bar{\Lambda}$ to determine $\varkappa(r)$ and hence the magnitude of the cross-sections.

Therefore,

$$V(r) \approx w\,\varrho(r) , \tag{39}$$

where w is the constant

$$w = -(2\pi\hbar^2/mk_{\text{lab}})G(\bar{\Lambda}) . \tag{40}$$

Corresponding to $G(\bar{\Lambda}) = G_1 + iG_2$, we have, of course, $w = w_1 + iw_2$, whence one may show that

$$\varkappa \approx -\tfrac{1}{2}\sqrt{2}(k_{\text{lab}}/E)w_2[1 + \tfrac{1}{2}(w_1/E)\,\varrho(r)]\,\varrho(r) . \tag{41}$$

Finally, the inclusion of the spin-orbit terms may be effected by replacing $G(\bar{\Lambda})$ by $G(\bar{\Lambda}) \pm iH(\bar{\Lambda})$ (see eqs. (29) and (30)).

In the detailed numerical calculations, the quantities $G(\Lambda)$ and $H(\Lambda)$ were taken from Section **2**. The constant's w_1 and w_2 were evaluated using values of $\bar{\Lambda}$ corresponding to $\theta = 5°$. The integral

$$(42) \qquad \mathrm{d}_{R,L}(\Lambda) = (4\pi/\Lambda)\int_0^\infty \mathrm{d}r\, r \sin \Lambda r\, \varrho(r) \exp[-(C-r)\,\varkappa(r)\,].$$

was then calculated using the Brown University IBM 650 computer, with $\varrho(r)$ and $\varkappa(r)$ as given by (20) and (41), respectively. The cross-sections obtained from eq. (34) are plotted in Figs. 4, 5, 6, 8 together with the original Born-approximation results. The polarizations, given by eqs. (32) and (33), are shown in Figs. 7, 9, 10, 11.

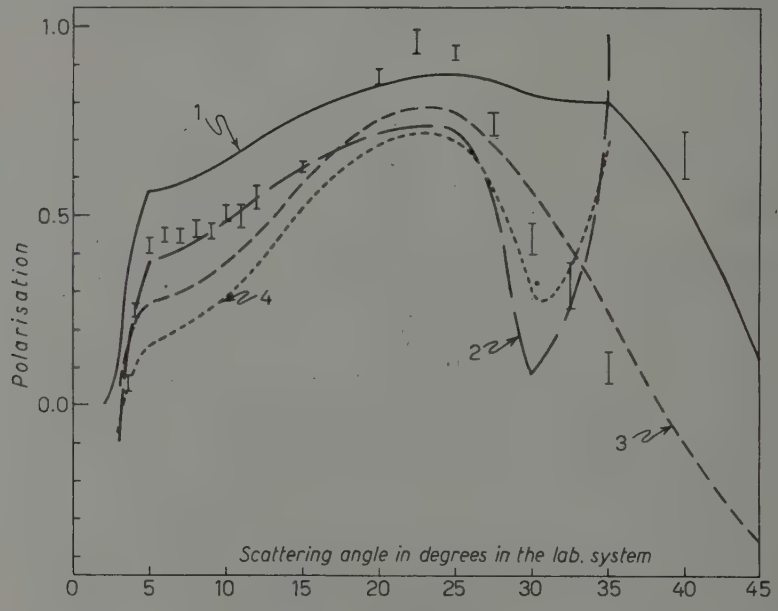


Fig. 9. – Proton-carbon polarization at 135 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (4.3) and (2.1) correspond to Gammel-Thaler and Signell-Marshak phase-shift sets, respectively; data from Dickson and Salter ([20]).

We notice that the absorption approximation not only decreases the cross-section by an approximately constant factor, but also moves the diffraction minima toward smaller angles, *i.e.*, it produces results which would correspond

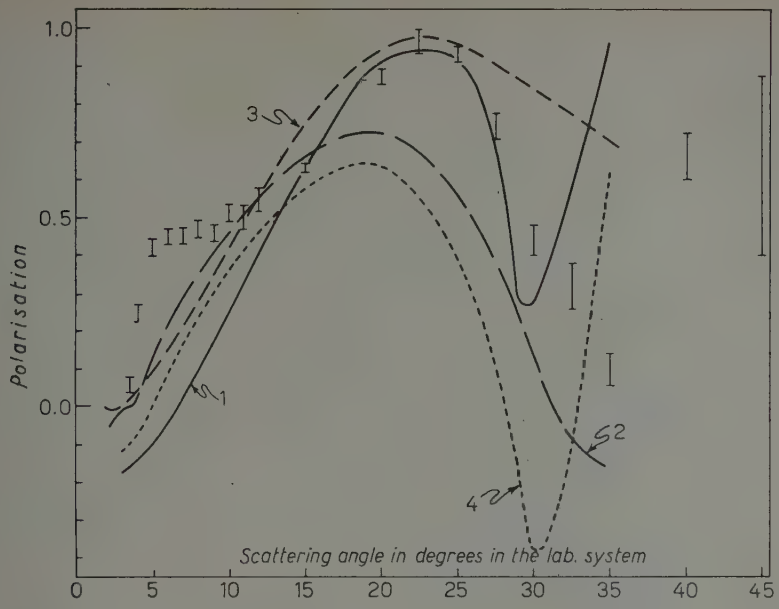


Fig. 10. – Proton-carbon polarization at 135 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (1.3) and (4.2) correspond to Ohnuma-Feldman phase-shift sets *E* and *L*, respectively; data from DICKSON and SALTER ([20]).

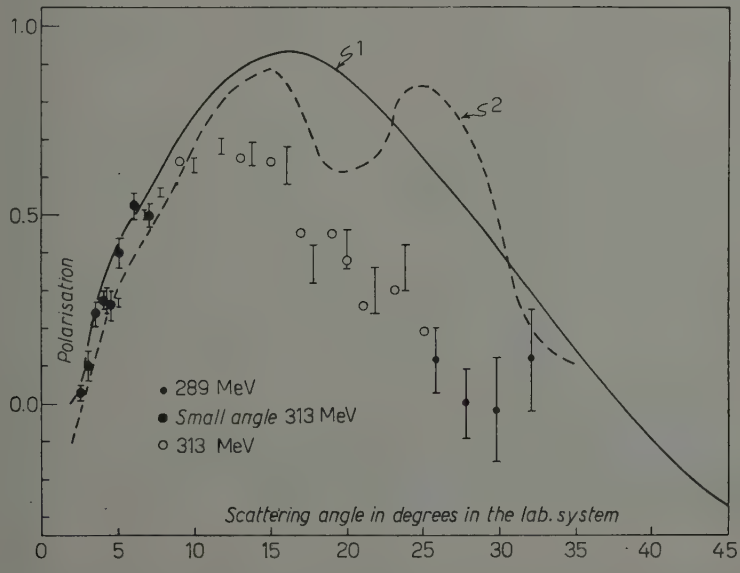


Fig. 11. – Proton-carbon polarization at 300 MeV in Born approximation, with and without absorption, including relativistic corrections; curves (2.1) correspond to Stapp phase-shift set no. 1; data from CHAMBERLAIN *et al.* ([4]).

to an increase of the nuclear radius. For the case of lead, the equivalent increase in radius would involve a factor of 1.23; for carbon the factor is 1.2. The resulting minima appear at the correct scattering angles for lead at 135 MeV and carbon at 300 MeV; for carbon at 135 MeV, a minimum is still predicted, contrary to observation. The absorption method also produces sharp dips in the polarization, minima which are absent in Born approximation, but which are evident in the carbon and lead polarization data at 135 MeV, and which are found in most of the exact solutions with simple assumed potentials ([19]). The absence of such a dip in the 300 MeV carbon data may be due to the presence of unresolved inelastic scattering at this energy.

It appears that the Ohnuma-Feldman phase-shift sets do not agree with the data, particularly in the Coulomb-interference region. The cross-sections (Fig. 5) indicate constructive interference, whereas the data show the sharp break characteristic of destructive interference. The predicted polarization (Fig. 10) is too small and rises too slowly in this same region. This is not unexpected, since these sets were derived before the very small-angle nucleon-nucleon data were available.

We can now distinguish between the carbon polarizations predicted by the Signell-Marshak and Gammel-Thaler phase-shift sets (Fig. 9). The latter are too small for all angles greater than 3°. The S-M polarization gives a good fit to the data for all angles less than 15°; it shows the proper shape at larger angles, though the maximum is smaller than that observed and the minimum is displaced toward smaller angles. At 300 MeV (Fig. 11), the Stapp phase shifts produce good agreement with the carbon polarization data for angles up to 10°.

The failure to obtain more than a qualitative fit to the angular distribution of the carbon polarization at larger angles can be understood in terms of the approximations used in this paper. The general behavior of the angular distribution of the scattering amplitudes is given by the modified form factors $\bar{d}(\Lambda)$ of (31) which are functions of the absorption factors $\varkappa$. These factors are determined, in turn, by the small-angle nucleon-nucleon amplitudes which, because of the optical theorem, are presumably well-determined for any phase-shift set. The precise angular distribution, however, is governed by the nucleon-nucleon scattering amplitudes at the momentum transfers in question; these amplitudes are characterized by energy transfers which differ from those of the free two-body system for the same momentum transfer and are hence approximated to an unknown degree by the two-body scattering phase shifts. Furthermore, the break-up into nucleon and nuclear parts implied by eq. (25)

([19]) See, for example, L. WOLFENSTEIN: *Ann. Rev. Nucl. Sci.*, **6**, 43 (1956) for discussion and additional references.

is valid only in the region where the two-body parameters vary only slightly with A.

The first diffraction minimum for lead occurs at a much smaller angle than for carbon ([20,21]). Hence, we do not expect the Born approximation to give as good a first approximation to the polarization in this case ([22]). Evidently the next approximation which takes absorption into account can also not be expected to give as good results for the polarization as for carbon. With the S-M phase shifts, the modified Born polarization allows no quantitative fit to the data (Fig. 7) at all—only a variation which is roughly the same as that predicted is observed.

The cross-sections are not so easily interpreted. That predicted by the Stapp phase shifts is a good quantitative fit at small angles, a fair fit at all angles (Fig. 6). The S-M and G-T cross-sections are similar to each other and to the carbon data (Fig. 4) but are a constant multiple ($\approx \frac{3}{2}$) of the data throughout most of the angular range. On the other hand, the S-M phase shifts give a lead cross-section (Fig. 8) which is a good quantitative fit for intermediate angles out to the first diffraction minimum, but fails at small angles where the data seem to indicate constructive Coulomb interference. Of course, it may well be that an improvement on the absorption approximation would resolve some of the discrepancies which have been noted.

4. – Conclusion.

In using the first Born approximation to calculate the effects of an impulse-approximation potential, we are considering only the scattering of the incident particle by a single nucleon in the nucleus; the effect of the remaining A-1 nucleons is purely kinematical. In this approximation, the polarization is the same for all spin-zero nuclei, while the cross-section is determined by the nucleon distribution within the nucleus. The modification of the first Born approximation to take into account absorption still deals with only a single scattering, but the wave function is damped by the imaginary part of the potential, *i.e.*, those multiple scatterings which are inelastic are included approximately.

In this manner, a solution is obtained to the nucleon-nucleus scattering problem which has many of the desired properties of the exact solution, *e.g.*, there is an increase in the apparent nuclear radius over that obtained from electron scattering which is usually attributed to the range of nuclear forces ([13]).

([20]) J. M. Dickson and D. C. Salter: *Nuovo Cimento*, **6**, 235 (1957).

([21]) J. M. Dickson: private communication.

([22]) See assumption five, p. 206, in Bethe's proof ([1]) of the Köhler-Levintov theorem.

It is easy to show ([23]) that the mean-square radius for a local optical-model potential which takes into account the angular variation of the nucleon-nucleon amplitude will be larger than that calculated with neglect of the angular variation by a term proportional to the derivative of the nucleon-nucleon amplitude in the forward direction with respect to the scattering angle, *i.e.*, by an amount which is interpreted as the « mean-square radius of the nucleon-nucleon force ». Such an increase is not seen in the first Born approximation.

The detailed shape and values of the cross-sections and polarizations are determined by the specific sets of phase shifts used. At angles sufficiently small so that off-energy-shell effects are small, they may give a means of differentiating among the different sets. Thus, in the small-angle range, it is found that sets E and L of Ohnuma-Feldman do not give a good fit either to the cross-section or the polarization for proton-carbon scattering at 135 MeV. The Gammel-Thaler phase shifts predict too little polarization at 135 MeV in carbon. Both the Signell-Marshak set and the phases due to Stapp give good agreement with the carbon polarization data, but only the latter yields a good representation of the carbon cross-section as well. It is, of course, to be emphasized that these conclusions are always subject to the limitations contained within the impulse approximation itself.

([23]) See, for example, KERMAN, MCMANUS and THALER: to be published.

RIASSUNTO (*)

Sono state calcolate, nell'approssimazione dell'impulso che utilizza i gruppi di spostamento di fase nucleone-nucleone di Signell-Marshak, Gammel-Thaler, Ohnuma-Feldman e Stapp, la sezione d'urto trasversale e la polarizzazione per lo scattering elastico di protoni su carbonio e piombo a 135 MeV e su carbonio a 300 MeV. Sono stati eseguiti i calcoli dettagliati sia nell'approssimazione ordinaria di Born che in funzione di una approssimazione modificata di Born nella quale si tenga conto dell'assorbimento dell'onda nucleonica diffusa elasticamente entro il nucleo. Sono stati valutati gli effetti relativistici, trovati peraltro di scarso rilievo. Si discute la possibilità di fare una distinzione fra i vari gruppi di spostamenti di fase nucleone-nucleone.

(*) *Traduzione a cura della Redazione.*

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The Decay of ^{44}Ti (*).

E. W. Cybulska and L. Marquez

Centro Brasileiro de Pesquisas Fisicas - Rio de Janeiro, Brasil

(ricevuto il 22 Giugno 1959)

Summary. — The decay of ^{44}Ti has been studied with scintillation spectrometers, coincidences and delayed coincidences. It was found that it decays by electron capture to the second excited state of ^{44}Sc. This is followed by a γ-ray of 79 keV and then a γ-ray of 70 keV. The 70 keV state is metastable with a half life of 0.18 μs.

1. – Introduction.

The nuclide ^{44}Ti was found by Sharp and Diamond ([1,2]) who described it as decaying by electron capture, followed by a γ-ray of 160 keV. They found that it decays only to the 3.9 h ground state of ^{44}Sc and that it does not populate the 2.4 d metastable state in ^{44}Sc. Their measurements indicated that the half life was greater than 23 years.

Huizenga and Wing ([3]) made a careful study of the γ-ray spectrum in the decay of ^{44}Ti, and proved that the decay is followed by two γ-rays in coincidence, one with an energy around 68 keV and the other with an energy around 76 keV, giving a composite peak at 72 keV. They found that the cross-over transition was less than 2% of the total number of desintegrations. They observed a peak at 144 keV, but by using Cu absorbers they came to the conclusion that it was all due to pile up, giving the above upper limit. They proposed a decay scheme.

We thought that it was worth while to investigate the decay of ^{44}Ti with the hope of measuring the energy of γ-rays precisely and to settle the question of the conflicting evidence about the γ-ray of the cross-over transition.

(*) Work sponsored in part by the Comissão Nacional de Energía Nuclear and the Conselho Nacional de Pesquísas of Brazil.

(1) R. A. Sharp and R. M. Diamond: *Phys. Rev.*, **93**, 358 (1954).

(2) R. A. Sharp and R. M. Diamond: *Phys. Rev.*, **96**, 1713 (1954).

(3) J. R. Huizenga and J. Wing: *Phys. Rev.*, **106**, 90 (1957).

There is also a practical reason why we should know the decay scheme of ^{44}Ti as well as possible. Since the half life of ^{44}Ti is long, it will be useful in Geophysical and Cosmophysical chronology, for instance, ^{44}Ti would be a good integrator for the cosmic ray flux received by an iron meteorite for periods of 1/10 of the half life to 4 half lifes before their fall.

The ^{44}Ti was prepared by irradiating a sample of 50 mg of Sc_2O_3 of 99% purity, the remaining 1% being mostly a mixture of the other rare earths. The Sc_2O_3 was mounted on a suitable target holder and irradiated with 22 MeV deuterons from the synchrocyclotron of the Comision Nacional de Energía Atomica, Buenos Aires. The integrated current that it received was of 100 μA·h. The reaction that produces ^{44}Ti is ^{45}Sc(d, 3n)^{44}Ti. The irradiated Sc was brought to our laboratory and allowed to decay for two months before processing.

2. – Experimental method.

The chemical procedure had to separate the ^{44}Ti from the ^{46}Sc and other rare earth impurities, from Zr and Hf, and from other smaller impurities. The Sc_2O_3 was fused with $HNaSO_4$, the melt was extracted with dilute HCl, 10 mg of Cu, 10 mg of Zn and 20 mg of Ti were added and the mixture of hydroxides was precipitated with NH_4OH. The hydroxides were dissolved in 0.1 N oxalic acid and the Ti separated from the Sc and the other metals that do not form complex salts with oxalic acid by means of an anion exchange column, following the technique of Walter ([4]). After separating the Sc, the Ti was eluted with 0.1 N HCl. It was concentrated and boiled with HNO_3 to destroy organic matter. It was evaporated to near dryness with H_2SO_4, 10 mg of Zr were added and the solution was poured over a cold mixture of NH_4OH and 20% H_2O_2. The precipitate was filtered and washed. The filtrate and washings were made acid with HCl and boiled thoroughly to destroy the H_2O_2. The Ti was precipitated with NH_4OH.

The precipitate was dissolved in H_2SO_4 and diluted to make it 20% by weight in H_2SO_4, 10 ml of H_2O_2 and 20 mg of Zr were added. The Zr was precipitated at 50 °C with 200 mg of secondary ammonium phosphate. The precipitate was filtered and washed with 5% $NaNO_3$. The filtrate was reduced with 4 g of Na_2SO_3 and neutralized with NaOH. The precipitate containing the Ti was filtered, washed and incinerated.

The precipitate was spread over a circle of 1.5 cm diameter, mounted between two plastic foils weighing 10 mg/cm^2 and then the measurements were carried out. We measured the γ-ray spectrum of the sample up to 800 keV.

([4]) R. I. WALTER: *Journ. Inorg. Nucl. Chem.*, **6**, 58 (1958).

In this region it showed peaks at 74 keV, 149 keV and the annihilation peak at 511 keV from the daughter ^{44}Sc. No other primary γ-ray could be detected and this showed that the sample was radiochemically pure. We saw of course the spectrum of the Compton effect from the 511 keV γ-ray and its back-scattered γ-ray of 170 keV.

We then made measurements of coincidences and delayed coincidences around the peak at 74 keV. We verified the result of HUIZENGA and WING, namely, there are two γ-rays in coincidence in the unresolved peak. We also found that the low energy γ-ray was delayed in regard to the high energy one. We fixed one of the γ-ray spectrometers to count on the low energy side of the composite peak and the other on the high energy side of it, and we were able to obtain the delayed coincidence curve shown in Fig. 1. A prompt coincidence curve taken with annihilation radiation is shown for comparison. This gives for the low energy γ-ray the half life of (0.18 ± 0.02) μs.

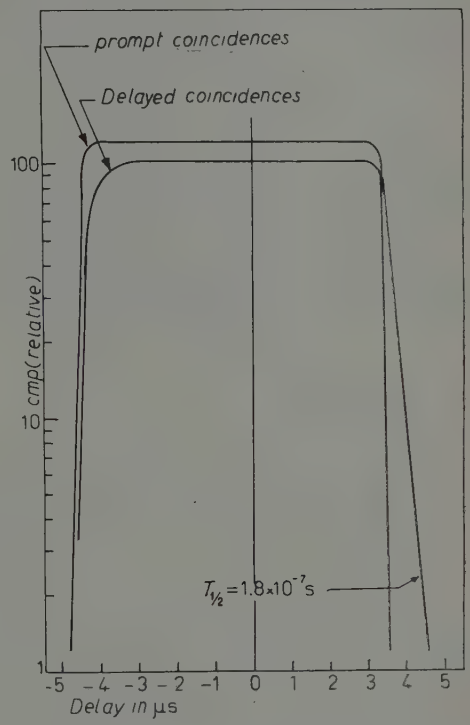

Fig. 1. – Prompt coincidence curve obtained with annihilation radiation. Delayed coincidence curve between the 79 keV γ-ray and the 70 keV γ-ray in ^{44}Ti. The delay is positive when inserted in the 79 keV channel and negative when inserted in the 70 keV channel.

Our delayed coincidence spectrometer used two single channel pulse height analyzers with NaI(Tl) crystals of $\frac{2}{3}$ in. diameter by 1 in. height. The pulses at the output of the analyzers were fed to the delay coincidence circuit. This was of conventional design. The smallest delay that could be introduced was 2.5 μs and the width of the coincidence channels was fixed at 2 μs. The time was measured with a Tektronix oscilloscope type 531. The instrument was checked by measuring the half life of the delayed state in ^{137}W and we found (0.6 ± 0.1) μs in agreement with other measurements ([5,6]).

([5]) D. E. BUNYAN, A. LUNDBY, A. H. WARD and D. WALKER: *Proc. Phys. Soc. (London)*, A **62**, 253 (1949).

([6]) S. DEBENEDETTI and F. K. MCGOWAN: *Phys. Rev.*, **74**, 728 (1948).

We could use the delay to measure the energy of the γ-rays precisely. This was done by introducing enough delay to cut the coincidence counting rate to half its value without delay while one of the spectrometers was set on the low energy side and the other was set on the high energy side of the composite peak. The spectrum was measured on one spectrometer and then on the other and gave the values of (70 ± 2) keV and (79 ± 2) keV. The delay was increased until the coincidence counting rate was one fourth of the value without delay. The measurements were repeated and gave the same results. The lines in these measurements were narrower and looked like single lines.

We carried out measurements on the peak at 149 keV with the aim of detecting the cross-over transition. We tried three techniques to reduce the contribution of the pile-up peak. The distance from the source to the detector was varied and Pb and Cu absorbers were used. It was not possible to detect it. An upper limit of 4 % relative to each of the γ-rays of 70 keV and 79 keV could be set. The upper limit could not be reduced because our source had only 20000 dpm and because of the back-scattered γ-ray of 170 keV.

3. – Discussion.

The half life of 0.18 μs found for the 70 keV transition allows us to classify it as $M1$. The half life given by the formula of Weisskopf (7) for the single particle transition is $8 \cdot 10^{-11}$ s, but experimentally measured half lives of $M1$ transitions sometimes are more than 1000 times slower than the values given by the single particle formula (7,8).

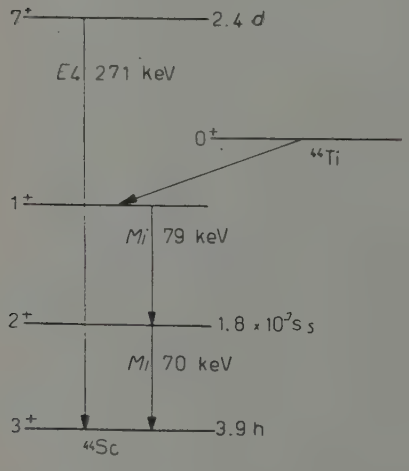

Fig. 2. – The proposed decay scheme of ^{44}Ti.

The spin and parity of ^{44}Ti is very probably 0^+, being an even-even nuclide. Its decay to the second excited state in ^{44}Sc is allowed, and its decay to the first excited state is forbidden. The decay of the ground state of ^{44}Sc to the ground state

(7) A. H. WAPSTRA, G. J. NIJGH and R. VAN LIESHOUT: *Nuclear Spectroscopy Tables* (Amsterdam, 1959), p. 71.

(8) M. GOLDHABER and A. W. SUNYAR: *Beta and Gamma Ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955), p. 467.

of ^{44}Ca is forbidden, and its decay to the first excited state is allowed. All this together with the information of this work leads to the assignment of 3^+, 2^+ and 1^+ for the ground state, first excited state, and second excited state of ^{44}Sc. The transition of 79 keV must be then $M1$ and the cross over transition $E2$.

All the information leads to the decay scheme shown in Fig. 2. This decay scheme is the same as the one of HUIZENGA and WING with some improvements. The energies of the γ-rays are known more precisely. Their order is known too. Finally, the existence of a metastable state in ^{44}Sc of 0.18 μs has been established. The 2.4 d-state is also shown in the decay scheme.

* * *

We are greatly indebted to the Comision Nacional de Energía Atomica, Buenos Aires, for making the irradiation for us without which this work could not have been carried out. We are also indebted to N. L. COSTA, I. G. ALMEIDA and M. LEVI for their help in the chemical separations, and to S. OSCHALINS for his help with the electronic equipment.

RIASSUNTO (*)

Il decadimento del ^{44}Ti è stato studiato con spettrometri a scintillazione, coincidenze e coincidenze ritardate. Si è trovato che decade per cattura di elettroni al secondo stato eccitato del ^{44}Sc. Al decadimento segue un raggio γ di 79 keV e poi un raggio γ di 70 keV. Lo stato a 70 keV è metastabile con una vita media di 0.18 μs.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIV, N. 3 1° Novembre 1959

A Method of Analysis of Events Involving Multiple Meson Production.

H. Huzita (*)

University of Wisconsin

(ricevuto il 23 Giugno 1959)

Summary. — A method is proposed for evaluating the primary energy in events which lead to multiple meson production. It is shown that, in principle, K-mesons could be distinguished from π-mesons if sufficient statistics were available. Several published events have been analyzed by this method and the results are compared with several theories.

1. – Introduction.

The phenomena of meson production in very high energy collision has been the subject of investigation by many authors. The amount of information derived from the study of the events available has been limited by the methods of analysis, rather than an insufficient number of events. One of the basic problems is an accurate determination of the primary energy. Two methods that are commonly employed to determine the primary energy are (1) the half angle method and (2) the logarithmic average of the angular distribution of the secondary particles from the interaction. In a previous work [1] it was shown that a better determination of the energy could be made if the energies as well as the angles of the secondary particles were used. However, it was assumed that

$$p_c \beta c = E_c \,,$$

where the subscript c denotes quantities in the center of mass system. The

(*) On leave of absence from St. Paul's University, Tokyo.

(1) H. Huzita: *Nuovo Cimento*, **6**, 841 (1957).

β was assumed to be the velocity of the center-of-mass system, rather than the velocity of the particle in the center-of-mass system.

In this paper the above assumption is not used, but rather the β of the particle in the center-of-mass system is used. Then appears a new term that is proportional to the square of the mass of the secondary particle and is of the same order of magnitude as the term containing the square of the angle. This term, which had not previously appeared, makes it possible to differentiate K-mesons from π-mesons at large angles. The assumptions made in the previous paper limit the applicability to very high energies and small angles and therefore are not applicable to energies attained by present day accelerators.

2. – Procedure.

The Lorentz transformation of the energy and the momentum from the laboratory system to a center-of-mass system that moves with velocity β is given by

$$E_c = \gamma(E_l - p_l\beta\cos\theta_l)\,, \tag{1}$$

$$p_c\cos\theta_c = \gamma(p_l\cos\theta_l - \beta E_l)\,, \tag{2}$$

$$p_c\sin\theta_c = p_l\sin\theta_l\,, \tag{3}$$

where

$$\gamma = 1/(1-\beta^2)^{\frac{1}{2}}$$

and θ is the angle between the direction of the secondary particles and the primary direction, c and l refer to the center-of-mass system and laboratory system respectively. The velocity of light is taken to be unity. Eliminating p_c from the equations (2) and (3) one gets the angular relation

$$\operatorname{tg}\theta_c = \frac{\operatorname{tg}\theta_l}{\gamma(1-\beta(E_l/p_l)(1/\cos\theta_l))}\,. \tag{4}$$

Using the nomenclature of GLASER *et al.* [2] for

$$\delta_E = 1-\beta\,,$$

$$\delta_{EZ} = \frac{E_l}{p_l} - 1\,,$$

$$\delta_\theta = 1-\cos\theta_l\,,$$

[2] R. G. GLASSER, D. M. HASKIN, M. SCHEIN and J. J. LORD: *Phys. Rev.*, **99**, 1555 (1955).

the equations (1) and (4) can be written as

$$E_c = \gamma E_l \left[1 - \frac{(1-\delta_\beta)(1-\delta_\theta)}{(1+\delta_E)}\right], \tag{5}$$

$$\operatorname{tg}\theta_c = \frac{\operatorname{tg}\theta_l}{\gamma(1-[(1-\delta_\beta)(1+\delta_E)/(1-\delta_\theta)])}. \tag{6}$$

Neglecting the terms of higher order than square of the deltas, the equations (5) and (6) become

$$E_c = \gamma E_c(\delta_\beta + \delta_E + \delta_\theta), \tag{7}$$

$$\operatorname{tg}\theta_c = \frac{\operatorname{tg}\theta_l}{\gamma(\delta_\beta - \delta_E - \delta_\theta)}. \tag{8}$$

These three small quantities can be approximated as

$$\delta_E \approx \frac{1}{2\gamma^2}, \tag{9}$$

$$\delta_E \approx \frac{1}{2}\left(\frac{m}{E_l}\right)^2, \tag{10}$$

$$\delta_\theta \approx \tfrac{1}{2}\theta_l^2, \tag{11}$$

which are usually satisfied in the experiments under consideration. For example, in order that the next terms of the Taylor expansions be one order smaller, the necessary conditions are

$$\begin{cases} \gamma \geqslant 2, \\ \dfrac{E_l}{m} \geqslant 3, \\ \theta_l \leqslant 1, \end{cases} \tag{12}$$

respectively. These conditions are almost always satisfied. Accepting these approximations, we have

$$E_c = \frac{1}{2}\gamma E_l\left(\frac{1}{\gamma^2} + \left(\frac{m}{E_l}\right)^2 + \theta_l^2\right), \tag{13}$$

or

$$\frac{1}{E_l} = \frac{1}{2E_c\gamma} + \frac{\gamma}{2E_c}\left[\left(\frac{m}{E_l}\right)^2 + \theta_l^2\right], \tag{14}$$

and

$$\operatorname{tg}\theta_c = \frac{2\operatorname{tg}\theta_l}{\gamma[(1/\gamma^2) - (m/E_l)^2 - \theta_l^2]}. \tag{15}$$

The equations (14) and (15) correspond to the equations (6) and (4) in the paper I. In the equation (14) $\operatorname{tg}\theta_l^2$ is replaced by $(m/E_l)^2+\theta_l^2$; however, if the secondary particles are assumed to be π-mesons, these values are known experimentally and the same analysis is possible. If experimental data of one event are plotted in the diagram $1/E_l$ *vs.* $(m/E_l)^2+\theta_l^2$ they distribute near a line, if E_c is nearly constant. From this line $1/2E\gamma$ and $\gamma/2E$ can be found and therefore γ and E as in paper I. If the particles are K-mesons and/or nucleons, the corresponding points in this diagram must be displaced to the left hand side of this line depending upon the assumption of their mass. In this case these points will be near the common line if the K-meson mass or the nucleon mass are assumed (see Fig. 1). In the small angle region these deviations are small so that it will be difficult to identify the particle.

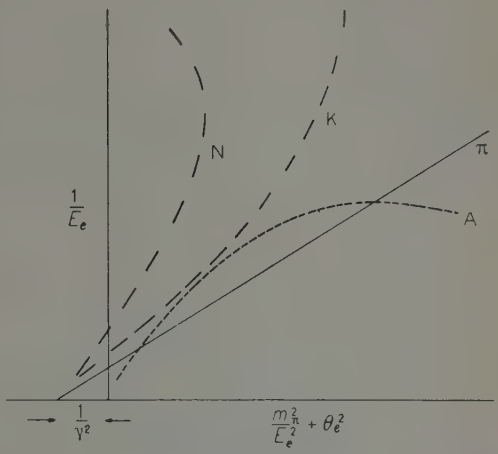

Fig. 1. – Schematic diagram of $1/E_l$ *vs.* $\theta_l^2+m_\pi^2/E_l^2$.

The inelasticity coefficient k, if all particles are π-mesons, is as in the paper I

$$k=\frac{1.5m_sE_c}{2(\gamma_c-1)M}, \tag{16}$$

where m_s is the number of the shower particles and M is the nucleon mass. If there are K-mesons the inelasticity becomes

$$k=\frac{(1.5n_\pi^\pm+2n_K^\pm)E_c}{2(\gamma_c-1)M}, \tag{17}$$

nevertheless the equation (16) is still a good approximation. If a sufficient number of events were available, with not too different primary energies, then in a plot of $1/\gamma E$ versus $\theta_l^2+(m/E_l)^2+(1/\gamma^2)$, the predicted lines for all the events would superpose.

3. – Discussion of the half angle method.

In the angular relation (15) the same term appears as in the energy relation. Putting $\theta_c=90°$ in the equation (15) one gets

$$\frac{1}{\gamma^2}=\left(\frac{m}{E_l}\right)^2+\theta_l^2. \tag{18}$$

If the term $(m/E_l)^2$ is negligible compared to θ_l^2, this relation becomes the same relation as the half angle equation.

$$\gamma = \frac{1}{\theta_l}.$$

However, the term $(m/E_l)^2$ is comparable to or frequently greater than θ_l^2. This is also true in the region near the half angle. This fact shows that the half angle method must overestimate in general the energy E_c, and underestimate the inelasticity k.

4. – Discussion of published events.

All of the data on the γ, E_c and k by this method are shown in the Table I. Also are given the values found in the original papers and those given by the method used in paper I.

TABLE I.

		γ	E	k
HOPPER *et al.* [3] (estimated by the method of paper I)	Author's analysis	23		0.1
	Paper I	46	0.3	0.03
	This paper	32	0.32	0.5
DEBENEDETTI *et al.* [4]	Author's analysis	$40 \div 50$		$0.66 \div 0.78$
	Paper I	46	1.0	0.4, 0.2
	This paper	22	1.1	~ 1
GLASSER *et al.*	Author's analysis	100	1.2	0.1
	This paper	100	1.3	0.2

The $1/E_i$ *vs.* $\theta_l^2 + (m_\pi^2/E_l^2)$ diagrams for each event are shown in Figs. 2, 3 and 4. The small scatter of the experimental points around the predicted line, in these figures is an indication that the assumption that the energy distribution of the secondary particles in the center-of-mass system is reasonably

[3] V. D. HOPPER, S. BISWAS and J. F. DARBY: *Phys. Rev.*, **84**, 457 (1951).

[4] A. DEBENEDETTI, C. M. GARELLI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, **4**, 1142 (1956).

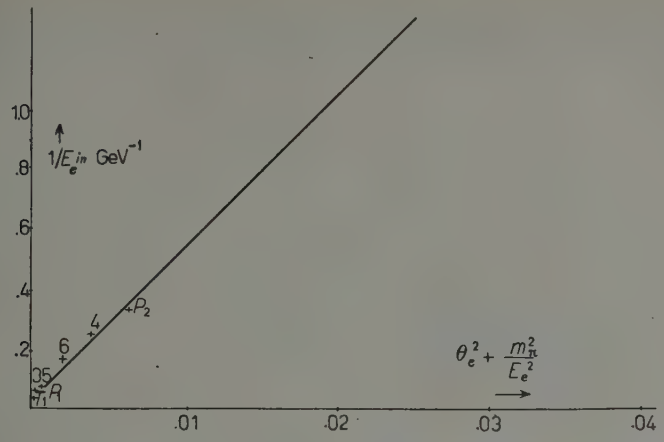

Fig. 2. – Event of Hopper *et al.* Number and letters by crosses are same as in their paper

narrow is reasonable. In the event of DEBENEDETTI *et al.*, some of the points lie outside of the main line. In fact if these points are replotted, assuming they are K-mesons, they then lie near the main line.

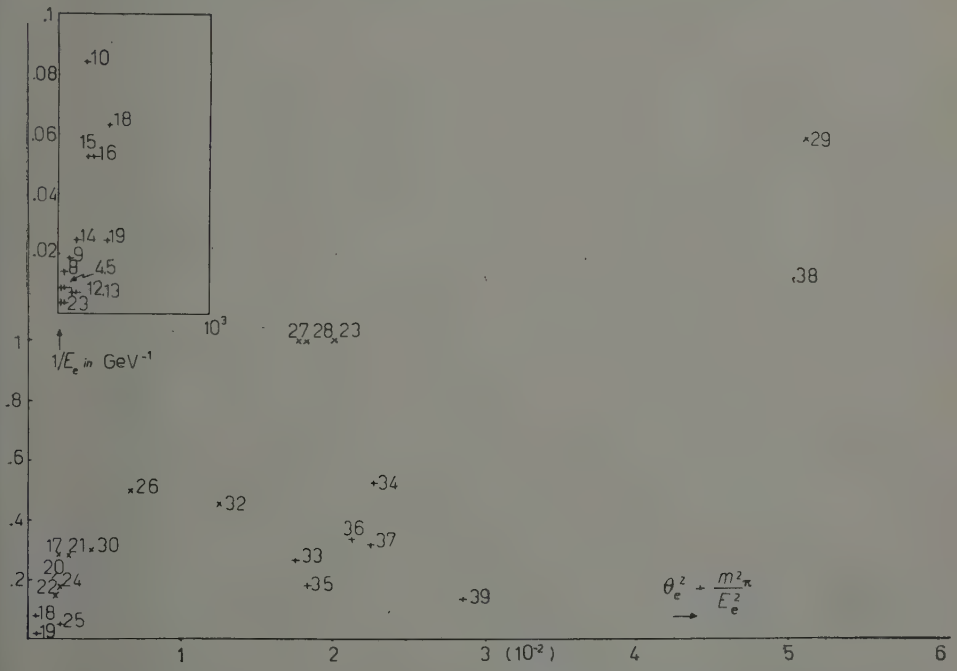

Fig. 3*a*. – Event of Debenedetti *et al.* Top square shows points near the origin of of the graph in the 10 times smaller scales of both axes. X is for a particle that is tentatively identified as K-meson.

It is not clear that all of these particles are indeed K-mesons, when it is remembered that the experimental errors are large and that the value E_e is

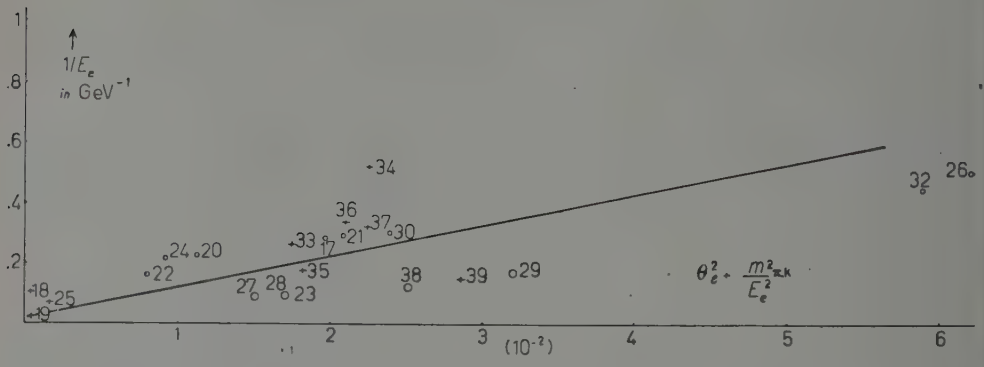

Fig. 3*b*. – Replot flom Fig. 3*a* assuming all X-points are K-mesons which are shown as 0). θ is 10 times changed the scale because of the scaling out.

an average energy and not the true energy of each particle. Assuming that these are K-mesons and that the other particles are π-mesons, the ratio of

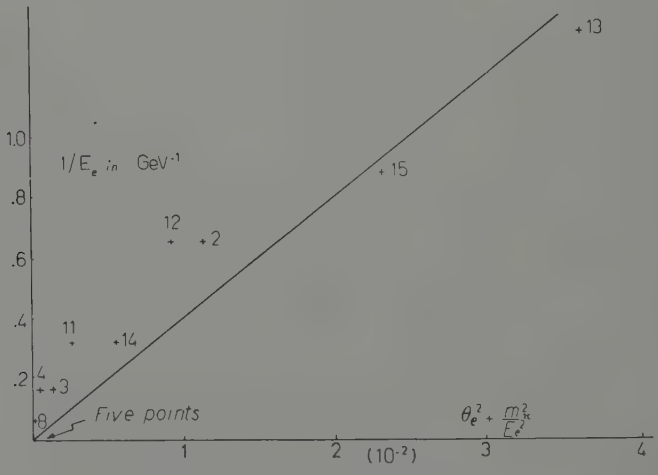

Fig. 4. – Event of Glasser *et al.*

K-mesons to all secondary particles is

$$\frac{2N_K^{\pm}}{2N_K^{\pm} + 1.5N_\pi^{\pm}} = \frac{20}{95} = 21\,\%\,.$$

This value is not inconsistent with the result ($20 \pm 7\%$) of EDWARD *et al.* [5], estimated by the observation of electron pairs from π^0 meson decays in emulsion.

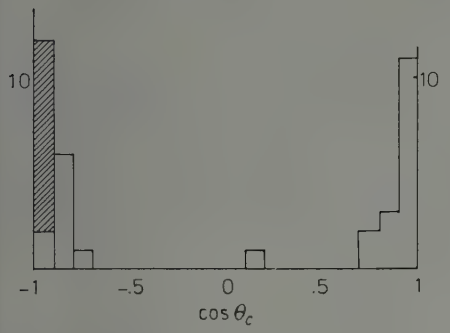

Fig. 5. – Angular distribution of the event of Debenedetti *et al.* The shaded part represents particles that seem as K-mesons.

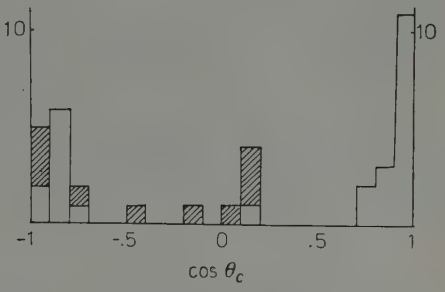

Fig. 6. – Angular distribution of the event of Debenedetti *et al.* The shaded part is recalculated assuming K-mesons.

In this method it is not necessary to invoke secondary interactions in the same nucleus to explain the deviations from a straight line, as in I.

Using the equation (15), the angular distribution in the center-of-mass system can be obtained. Fig. 5 is drawn assuming that all the secondary particles are π-mesons. In Fig. 6 it is assumed that all points which lie at large distance from the main line are due to K-mesons. It can be seen that the latter assumption results in a more nearly symmetrical distribution.

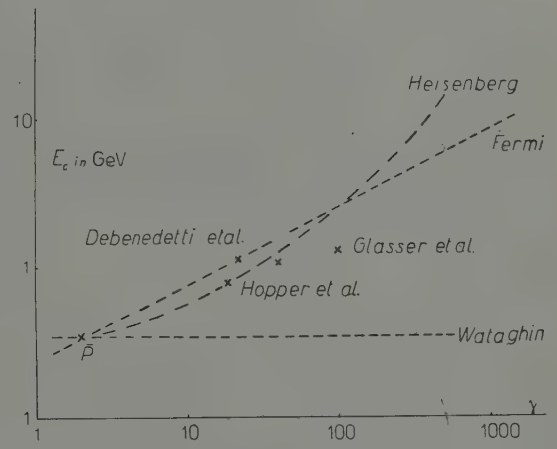

Fig. 7. – E_c *vs.* γ diagram. $\bar{p}$ shows the data of antiprotons at rest.

If there is an angular dependence of the Energy Spectrum of the secondary particles it will be visible in this diagram. If the average energy of the secondary particles with small θ is higher than that of particles with large θ, then the distribution of points will appear to be on

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[5] B. EDWARDS, J. LOSTY, D. H. PERKINS, K. PINKAU, J. REYNOLDS: *Phil. Mag.*, **3**, 237 (1958).

a convex line near the main line. If the converse is true, the points will appear to lie on a concave line.

In each of the three events shown in Fig. 2, 3 and 4 there is a tendency for the points to lie on a convex line. This is also true for many events reported by SCHEIN at the Padua-Venice Conference in 1957.

Fig. 7 shows a plot of the average energy of the secondary particles in the center-of-mass system, E_c as a function of the primary energy in the same system. In spite of the small number of events, it would seem that E_c increases slowly with increasing γ. In Fig. 7 are also shown the results of various theories, normalized to the antiproton point, which would seem to be the most reliable.

* * *

Discussions with Professor N. DALLAPORTA, Drs. M. BALDO-CEOLIN, U. CAMERINI, and W. F. FRY were helpful and stimulating. Corrections suggested by S. MATSUMOTO are gratefully acknowledged.

RIASSUNTO (*)

Si propone un metodo per calcolare l'energia primaria in eventi che portano ad una produzione mesonica multipla. Se si riconosce valida una teoria statistica adeguata, si dimostra che, in linea di principio, i mesoni K possono essere distinti dai mesoni π. Con questo metodo sono stati analizzati numerosi eventi già pubblicati; si confrontano i risultati con varie teorie.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIV, N. 3 1° Novembre 1959

The Nature of the Neutral Particles Emitted in $K_{\beta 3}$ Decay.

K. Imaeda and M. A. Shaukat (*)

School of Cosmic Physics, Dublin Institute for Advanced Studies - Dublin

(ricevuto il 30 Giugno 1959)

Summary. — An event has been observed in photographic emulsion in which a K^+-meson appears to decay at rest to a positron and a Dalitz electron-pair. The results of dynamical analysis, coupled with considerations of conservation of spin are interpreted as providing direct evidence for the emission of π^0 in $K_{\beta 3}$ decay.

During the course of a systematic study [1], using stacks exposed at the Berkeley Bevatron, of electron-pairs associated with K^+ decays, an event has been observed in which the tracks of three associated charged particles have been identified as being due to electrons. A sketch of this event is shown in Fig. 1. The primary particle K^+ comes to rest at *P*. The estimate of the mass of K^+, ob-

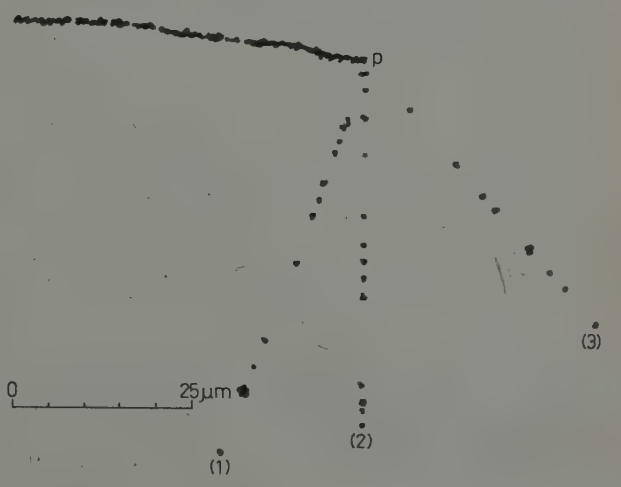

Fig. 1. – The decay of K^+-meson at rest into an electron pair and a positron.

(*) On leave of absence from the Panjab University, Lahore, Pakistan.

(1) G. Alexander, R. H. W. Johnston and C. O'Ceallaigh: *Nuovo Cimento*, **6**, 478 (1957).

tained using the constant sagitta method, was $(1020 \pm 140)\ m_e$. Ionization measurements, using the techniques of « blobs and holes » and also of gap-counts, confirm this result, and demonstrate that the primary particle had come to rest or was moving with negligible velocity. The three lightly ionizing particles (1, 2 and 3) proceeding from the point of arrest, were all reasonably flat so that reliable measurements of scattering and ionization could be made upon them. Details of the configuration of these tracks, and of the measurements are given in Table I. Each value given is the mean of the results of two independent observers.

TABLE I.

Track no.	Dip (unprocessed)	Projected angles	Space angles	Blob-density	$p\beta c$ (MeV)	Measured segment from P.
1	$+20 \pm 1°$	0°	$\widehat{1,2} = 36°$	1.10 ± 0.08	8 ± 2	$(0 \div 1.3)$ mm
2	$-10 \pm 0.5°$	$18.5 \pm 0.5°$	$\widehat{2,3} = 55°$	1.13 ± 0.03	166 ± 31	$(0 \div 16)$ mm
3	$+27 \pm 1°$	$60.0 \pm 0.5°$	$\widehat{3,1} = 56°$	1.13 ± 0.06	24 ± 5	$(0 \div 2.6)$ mm

The estimates of blob-density are all referred to the minimum of the ionization curve, and are based on calibrations carried out in the same plates in previous investigations ([1,2]).

Scrutiny of Table I will show that, in the case of tracks 1 and 3, the ionization and scattering measurements are such that they are unambiguously identified as being due to electrons. On the basis of the estimates of $p\beta c = (166 \pm 31)$ MeV and $\bar{b} = 1.13 \pm 0.05$ made on the first segment of length 4 mm of track 2, it is not possible to exclude the possibility that the track is that of a π-meson and that, in fact, we are dealing with a case of $K_{\pi 2}$-decay accompanied by a Dalitz pair. However, when the track is traced into 3 successive strips and scattering measurements are continued, it emerges very clearly that the track is that of an electron. The experimental figures are contained in Table II.

Since clearly, the validity of this conclusion rests almost entirely on the absence of error in following the track through the stack, the associations were carefully verified, the track being followed in both senses by five observers from our group. Furthermore, the dynamical analysis excludes the possibility that the event is $K_{\pi 2}$.

The event is susceptible to dynamical analysis using the procedure followed by YEKUTIELI *et al.* ([3]) in their study of a case of $K_{\mu 3}$-decay associated with

([2]) G. ALEXANDER and R. H. W. JOHNSTON: *Nuovo Cimento*, **5**, 363 (1957).
([3]) G. YEKUTIELI, M. F. KAPLON and T. F. HOANG: *Phys. Rev.*, **101**, 506 (1956).

TABLE II.

Distance from P of mid-point of segment (mm)	1.5	4.8	8.0	11.3	14.3
No. of cells (150 μm)	20	19	19	20	21
$p\beta c$ (MeV)	162±36	94±21	67±15	61±13.5	49±11
$\bar{b}$	1.13±0.05	1.12±0.05	—	1.13±0.05	

a Dalitz pair. For this purpose, we may consider the electron-pair to be replaced by an equivalent particle having momentum P_0, total energy E_0 and mass M_0 given by

$$M_0^2c^4 = E_0^2 - P_0^2c^2 , \tag{1}$$

where $P_0^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\theta_{12}$, the subscripts 1 and 2 referring to the momenta of the individual electrons, θ_{12} being the opening angle of the pair.

If we consider all three electrons to have been produced directly in K-decay, $K^+ \to e^+ + e^- + e^+ + X$, we may endeavour to place limits on the mass of X, assuming one standard deviation for the uncertainty of determination of the momenta of the electrons. We find $168 \leqslant M_X c^2 \leqslant 286$ MeV. In contrast however, to the case of YEKUTIELI *et al.* [3], the possibility that the mass of X corresponds to that of a single known particle cannot be excluded by such arguments, but the dynamical possibility $X = \pi^0$ may be rejected immediately from considerations of spin.

It appears more reasonable to suppose that the electron pair arises from the direct decay of a π^0-meson following the process $\pi^0 \to e^+ + e^- + \gamma$, and that spin-conservation requires the emission of a second neutral particle say Y, the decay process of the K^+ being represented by

$$K^+ \to e^+ + \pi^0 + Y \to e^+ + (e^+ + e^- + \gamma) + Y .$$

Limits may now be set to the mass of Y using arguments analogous to those of YEKUTIELI *et al.* We may compute the value of the momentum of the π^0 referred to the rest system of the K^+ in terms of the momentum P_0 and mass M_0 of the particle equivalent to the electron-pair and the angle θ between their momenta defined by

$$\cos\theta = \frac{\boldsymbol{P}_{\pi^0} \cdot \boldsymbol{P}_0}{P_{\pi^0} P_0}, \tag{2}$$

We have

$$\gamma(1-\beta\beta_0\cos\theta)=\frac{M_{\pi^0}^2c^4+M_0^2c^4}{2M_{\pi^0}c^2E_0}=A\,, \tag{3}$$

where

$$\beta=\frac{P_{\pi^0}c}{E_{\pi^0}}\,,\qquad \beta_0=\frac{P_0c}{E_0}\,.$$

From the above we may find an expression for the momentum of the π^0-meson in terms of $\beta_0\cos\theta$ in the rest system of the decaying K^+

$$P_{\pi^0}=\frac{A\beta_0\cos\theta\pm(\beta_0^2\cos^2\theta+A^2-1)^{\frac{1}{2}}}{1-\beta_0^2\cos^2\theta}\cdot M_{\pi^0}c\,. \tag{4}$$

We may set limits on the mass of the neutral particle Y by combining (4) with (5)

$$M_Y^2c^4=(M_Kc^2-E_s-E_{\pi_0})^2-|\boldsymbol{P}_sc+\boldsymbol{P}_{\pi_0}c|^2\,, \tag{5}$$

where E_s and $\boldsymbol{P}_s$ stand for the total energy and momentum of the charged secondary particle.

A priori, it is not possible to determine which electron is the charged secondary particle. We consider therefore the three possibilities in turn, computing the value of the parameter A and setting out the results together with the computed limits on the mass M_Y of the neutral particle Y.

TABLE III.

Case	Assumed secondary $p\beta c$ (MeV)	Assumed electron pair		M_0 (MeV)	β_0	A	Limits on M_Y (MeV)
		E_0 (MeV)	P_0c (MeV)				
(*a*)	166±31	32.0±5.4	29.3±5.4	12.8±1.2	0.916	2.12 ±0.29	$128\geqslant M_Y\geqslant 0$
(*b*)	8± 2	190±31	181±31	592±4.0	0.950	0.424±0.062	$221\geqslant M_Y\geqslant 0$
(*c*)	24± 5	174±31	173±31	22.5±2.0	0.992	0.399±0.058	$217\geqslant M_Y\geqslant 0$

The fact that for case (*a*) the value of $A>1$ precludes application of the analysis of YEKUTIELI *et al.* and we have arrived at the limits for M_Y in the following manner. The set of curves shown in Fig. 2 show the values of certain dynamical parameters all plotted as a function of the momentum P_{π^0}. Curve I gives the corresponding values of θ defined as in (2). For this case (*a*), the

calculated value of the analogous angle between the assumed charged secondary and $\boldsymbol{P}_0$ is 45.5°. Hence, the angle between the π^0 momentum vector and that of the assumed secondary may take on any value between the limits $\theta \pm 45°.5$. The values of the extremes of the magnitude of the resultant mo-

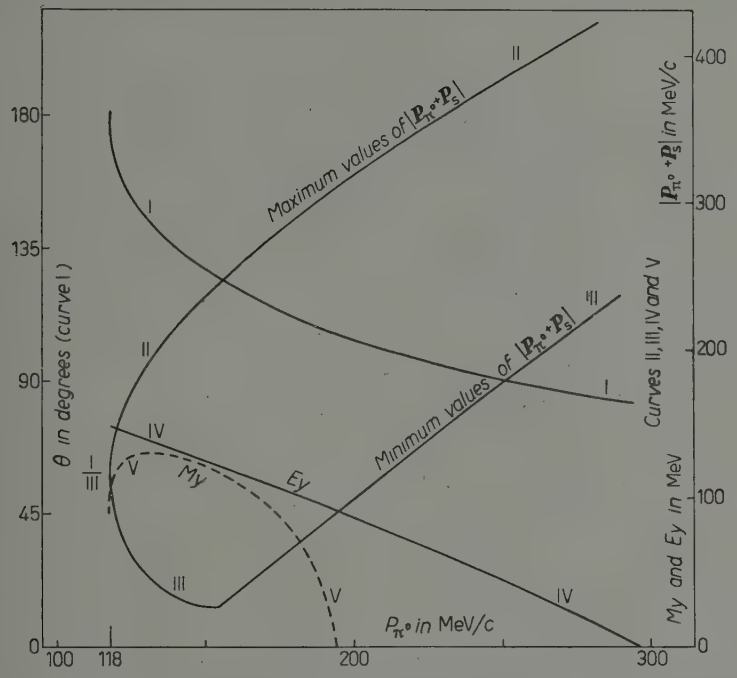

Fig. 2. – Curve I shows the relation between the momentum of the π^0-meson, P_{π^0}, and θ (defined by equation (2)) as obtained from expression (4). Curves II and III show the maximum and minimum values of $|\boldsymbol{P}_s+\boldsymbol{P}_{\pi_0}|$, the resultant momentum of the secondary and the π^0-meson. Curve IV shows the total energy of the neutral particle Y in the three body decay mode $K^+ \rightarrow e^+ + \pi^0 + Y$. Curve V shows the rest mass of the neutral particle Y, as calculated from equation (5) using minimum values of $|\boldsymbol{P}_{\pi^0}+\boldsymbol{P}_s|$.

mentum $\boldsymbol{P}_{\pi^0}+\boldsymbol{P}_s$ which correspond to these angular limits are plotted as Curves II and III. Since the K^+ meson decays at rest, these values are also the extreme values of the momentum of the neutral particle Y. The total energy E_Y is plotted as Curve IV; values for the upper limit of M_Y obtained from (5) are shown as Curve V. The lower limit of M_Y is zero. Thus, from this kinematical analysis, confining ourselves to neutral particles of which the existence has been established or conjectured the particle Y could be π^0, γ, μ^0 or ν. We may reject π^0 and γ because of the necessity of conservation of spin. Since the existence of μ^0 has not yet been established, the conclusion

that the three-body K_β decay scheme is

$$K_{\beta 3} \to e^+ + \pi^0 + \nu \to e^+ + (e^+ + e^- + \gamma) + \nu$$

is consistent with spin conservation and with our experimental measurements.

* * *

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RIASSUNTO (*)

Si è osservato in emulsione fotografica un evento in cui un mesone K^+ decade a riposo in un positrone ed una coppia di elettroni Dalitz. Dai risultati dell'analisi dinamica, congiunti a considerazioni sulla conservazione dello spin, si trae la prova immediata dell'emissione di un π^0 nel decadimento $K_{\beta 3}$.

(*) *Traduzione a cura della Redazione.*

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On the Pionic Decay of the Λ-Particle.

L. TENAGLIA

Istituto di Fisica dell'Università - Bari

(ricevuto il 2 Luglio 1959)

Summary. — On the hypothesis that the pionic decay of the Λ-particle is due to a Fermi interaction, the corrections to the axial an vector coupling constants are evaluated by the Goldberger and Treiman procedure. On account of the fact that only the intermediate nucleon-antinucleon states give appreciable contribution to the corrections, their evaluation depends on the complex phase shifts of the nucleon-antinucleon scattering. Such phase shifts are not known enough to be able to derive from them good information on the pionic Λ-decay; therefore—aside from a qualitative discussion of the problem based on the complex phase shifts of the nucleon-antinucleon scattering—we refer explicitly to the well known corrections to the axial coupling constant in the decay of the pion ($\pi \to \mu\nu$). This comparison allows to apply quantitatively the procedure only to the decay $\Lambda \to p\pi^-$; the results are satisfactory. It is also possible to verify that the constants in the decay hamiltonian of the Λ are strongly modified when one takes into account the possible corrections to the Fermi interactions amongst bare particles (virtual emission of pions by the Λ-particle, and reabsorption by the Λ or by the nucleon in the final state). Therefore, with the followed procedure, one has to assume that the Fermi interaction operates among dressed spinors.

The pionic decay of the hyperons has been proposed by many authors [1] to be a consequence of a Fermi interaction.

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[1] C. FRANZINETTI and G. MORPURGO: *Suppl. Nuovo Cimento*, **6**, 470 (1957), p. 702; M. GELL-MANN and H. A. ROSENFELD: *Annual Rev. of Nuclear Science*, **7**, 406 (1957); S. B. TREIMAN, 1958 *Annual International Conference on High Energy Physics at CERN* (CERN, Geneve, 1958), p. 276.

The problem of the direct evaluation of the corrections to the Fermi interaction was also studied by S. OKUBO, R. E. MARSHAK and E. C. G. SUDARSHAN: *Phys. Rev.*, **113**, 944 (1959). The mainly conclusion of such authors concerns the possibility of a $\Delta I = \frac{1}{2}$ plus $\Delta I = \frac{3}{2}$ decay.

In the following calculation, such a type of interaction has been considered in connection to the pionic decay of the Λ-particle ($\Lambda \to \mathcal{N}\pi$). The Goldberger and Treiman [2] procedure has been applied to the evaluation of the effective parameters of the Fermi interaction.

The examined decay process is very similar, with respect to its structure (that is, its graphs), to the pion decay ($(\pi^{\pm} \to \mu^{\pm}\nu)$, also [3] studied by Goldberger and Treiman. The difference consists in the presence (in the actual case) of a vector interaction term, in the intermediate state, and in the different importance of the correction due to secondary interaction.

The most remarkable point of the comparison is the following: the characteristic parameters for the protonic decay of the Λ-particle ($\Lambda \to p\pi^-$) can be deduced from the G.T. data and from other phenomena sufficiently known. On the contrary, it is *a priori* impossible to deduce, by the same procedure, the characteristic parameters of the nucleonic decay of the Λ-particle ($\Lambda \to \mathcal{N}\pi^0$).

However, also in this case we have enough data to be able to say that the G. and T. procedure can explain the process and give results in agreement with the experimental data.

The matrix element for the $\Lambda \to \mathcal{N}\pi$ decay, following the usual procedure by Lehmann *et al.* [4], is given by:

$$(1) \qquad \langle N\pi_{\text{out}} | \Lambda \rangle = i(2\omega_q)^{-\frac{1}{2}} \int \mathrm{d}^4x \langle \pi | \varphi^*(x) | 0 \rangle \langle N | I(x) | \Lambda \rangle = \\ = i(2\pi)^4 \delta(p_{\mathcal{N}} + p_\pi - p_\Lambda) \langle N | I(0) | \Lambda \rangle .$$

$$(2) \qquad \langle N | I(0) | \Lambda \rangle = i \left(\frac{P_{0\Lambda}}{M_\Lambda} \right)^{\frac{1}{2}} \tilde{u}_{\mathcal{N}} \int \mathrm{d}^4x \exp[i(p_\Lambda - p_{\mathcal{N}})x] \langle 0 | (\tilde{f}(x), I(0))_- | \Lambda \rangle = \\ = i \left(\frac{p_{0\Lambda}}{M_\Lambda} \right)^{\frac{1}{2}} \tilde{u}_{\mathcal{N}} \int \mathrm{d}^4x \exp[i(p_\Lambda - p_{\mathcal{N}})x] \theta(-x) \langle 0 | [\tilde{f}(x), I(0)] | \Lambda \rangle .$$

In the expression (1) and (2) there appear, besides the indices corresponding to the eigenstates of the nucleon $\mathcal{N}$, of the Λ-particle and of the pion π (ω_q stands for the pion energy in the final state), also the densities of the source for the pion field, $I(x)$, and for the spinor field, $f(x)$. In the evaluation

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[2] M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **11**, 354 (1958): in the following G.T.

[3] M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **110**, 1178 (1958): in the following G.T. II.

[4] H. P. Lehman: *Course on Mathematical Problems of Quantum Theory* (Varenna, 1958) and bibliography cited htere.

of the above matrix element two types of interactions, which might have a certain influence, are disregarded, for the moment:

a) The pion-nucleon scattering in the final state.

b) The interaction due to emission of quanta (pions or K-mesons) from the initial Λ, and their subsequent absorption by the Λ itself or by the final nucleon.

From invariance arguments and by assuming that the $f(x)$ source derives from a $V-A$ Fermi interaction, the expression (2), can be written as follows:

$$\tilde{u}_{\mathcal{N}}\{F_1\gamma_\lambda(p_\Lambda - p_{\mathcal{N}})_\lambda + F_2\gamma_\lambda\gamma_5(p_\Lambda - p_{\mathcal{N}})_\lambda\}u_\Lambda\,, \tag{2'}$$

where F_1 and F_2 are functions of the (invariant) quantity $(p_\Lambda - p_{\mathcal{N}})^2 = p_\pi^2$, and can be evaluated from dispersion relations.

With the hypothesis that the two nucleons responsible with $\mathcal{N}$ and Λ for the Fermi interaction, are a nucleon-antinucleon pair $\mathcal{N}\overline{\mathcal{N}}$, the absorption parts of F_1 and F_2 are:

$$A_i = \pi\left(\frac{p_{0\Lambda}}{M_\Lambda}\right)^{\frac{1}{2}}\tilde{u}_{\mathcal{N}}\langle 0\,|\,I\,|\,N\bar{N}\rangle\langle N\bar{N}\,|\,\tilde{f}_i\,|\,\Lambda\rangle\,, \tag{3}$$

with

$$\begin{cases} \langle N\bar{N}\,|\,\tilde{f}_1\,|\,\Lambda\rangle = \dfrac{M_\Lambda}{(n_0\bar{n}_0)^{\frac{1}{2}}}\,\bar{u}_{\mathcal{N}}(ia\gamma_\lambda\gamma_5 - b[n+\bar{n}]_\lambda)v_{\mathcal{N}}\cdot i\gamma_\lambda\gamma_5 u_\Lambda\,, \\ \langle N\bar{N}\,|\,\tilde{f}_2\,|\,\Lambda\rangle = \dfrac{M_\Lambda}{(n_0\bar{n}_0)^{\frac{1}{2}}}\,\tilde{u}_{\overline{\mathcal{N}}}(c\gamma_\lambda - id\sigma_{\lambda\mu}[n+\bar{n}])u_{\mathcal{N}}\cdot\gamma_\lambda u_\Lambda\,, \end{cases} \tag{4}$$

and with

$$\tilde{u}_{\mathcal{N}}\langle 0\,|\,I\,|\,n\bar{n}\rangle = \frac{iM_\Lambda}{(n_0\bar{n}_0)^{\frac{1}{2}}}\,K([n+\bar{n}]^2)\tilde{u}_{\mathcal{N}}\cdot\tilde{u}_{\mathcal{N}}\gamma_5 u_{\mathcal{N}}\,, \tag{4'}$$

a, b, c, d represent some functions of the energy-momentum $\xi = [n+\bar{n}]^2$, and have been evaluated, together with $K(\xi)$, by Goldberger and Treiman.

Direct method for the evaluation of a, b and K for the $\Lambda \to p\pi^-$ decay at low energies, can be deduced from the knowledge of the $\mathcal{N}\overline{\mathcal{N}}$ scattering complex phase shifts. The scattering cross-sections for such particles are known, in fact, at low energy with a sufficient approximation. Their evaluation, by using a phenomenological potential (in a Born approximation and by the phase shift method) and different values of the parameters, has been carried out [5] by Chew-Ball and [6] by Jancovici, Gourdin and Verlet. Using

[5] J. S. Ball and G. F. Chew: *Phys. Rev.*, **109**, 1385 (1958).

[6] M. Gourdin, B. Jancovici and P. Verlet: *Nuovo Cimento*, **8**, 485 (1958).

the data of the latter authors, the values of $a(\xi)$, $b(\xi)$ and $K(\xi)$, has been re-evaluated at $\xi = -\omega_q^2$; we have verified that the integral J from following form. (5′) and from G.T. II, form. (25), assumes a value which is in good agreement with the value determined by G.T. II ((eq. (27)).

This check has only an orientative character and it is convenient to deduce $K(-\omega_q^2)$ and $b(-\omega_q^2)[\approx K(-m_\pi^2), b(-m_\pi^2)]$ directly from the experimental data of the $\pi \to \mu\nu$ decay.

Then one gets, disregarding negligible terms:

$$(5) \qquad F_1 \simeq -\frac{M\sqrt{2}}{2\pi^2}\,\frac{GJg_A}{1+G^2J/2\pi^2}\,, \qquad F_2 \simeq \frac{M\sqrt{2}}{2\pi^2}\,JGg_V\,,$$

M is the mass of the nucleon, g_A and g_V the coupling constants of the axial and vector Fermi interactions, G the coupling constant of the pion-nucleon interaction; J stands for the integral defined by G.T. II, form. (25), for $\pi \to \mu\nu$, and which, in the present case, assumes the form:

$$(5') \qquad J = \frac{1}{2}\int_{4M^2}^{\infty} \frac{\mathrm{d}\xi}{\xi-\omega_q^2}\left(\frac{\xi+4M^2}{\xi}\right)^{\frac{1}{4}} \cos\varphi(\xi)\,\exp\left\{\frac{\xi-\omega_q^2}{\pi}\int_{4M^2}^{\infty}\frac{\varphi(y)\,\mathrm{d}y}{(y-\xi)\,(y-\omega_q^2)}\right\}.$$

In the (5′), $\varphi(\xi)$ is the function of the complex phase shifts for the $\mathcal{N}\overline{\mathcal{N}}$ scattering, defined by G.T. II, eq. (29).

The direct evaluation of $b(\zeta)$, $K(\zeta)$, and $J(\zeta)$ for $\zeta = -\omega_q^2$ ($= -1.226\, m_\pi^2$ in the present case, in the system in which the Λ-particle is fixed; a, b and K for $\zeta = -m_\pi^2$ assume practically the same value) has been carried out essentially in order to clarify qualitatively the problem of the difference between the probabilities of the two decays

$$(6) \qquad \Lambda \to \mathrm{p}\pi^-\,, \qquad \Lambda \to \mathrm{n}\pi^0\,,$$

which, evidently, depend also on the different corrections to the constants of the Fermi interaction, and therefore, through the integral J of eq. (5′), on the different value of the complex phase shifts for the intermediate nucleon and antinucleons.

First of all one can separate in the preceding matrix elements, the isotopic spin I eigenfunctions of the initial and final states:

$$(7) \qquad \chi_{\mathrm{in}} = \chi_0^0 = \Lambda\,, \qquad \chi_{\mathrm{fin}} = \chi_{\frac{3}{2}}^{-\frac{1}{2}} = \sqrt{\tfrac{2}{3}}\,n\pi^0 + \sqrt{\tfrac{1}{3}}\,p\pi^-\,, \qquad (I=\tfrac{3}{2})\,,$$
$$= \chi_{\frac{1}{2}}^{-\frac{1}{2}} = -\sqrt{\tfrac{1}{3}}\,n\pi^0 + \sqrt{\tfrac{2}{3}}\,p\pi^-\,. \qquad (I=\tfrac{1}{2})\,.$$

Usually the difference between the decay probabilities of the two proces-

ses (6) is deduced from the difference between the corresponding factors in $\chi_{\frac{1}{2}}^{-\frac{1}{2}}$, assuming that the relation $I = \frac{1}{2}$ holds in the decay. But in the Fermi theory of the decay, a difference comes out also from the presence of different $\mathcal{N}\overline{\mathcal{N}}$ intermediate states. In fact, for the two decays (6) one gets (7) respectively:

(8) $\quad \mathcal{N}\overline{\mathcal{N}} = -\,\mathrm{n}\bar{\mathrm{p}} \qquad I = 0 \qquad I_3 = -1$

and

(8′) $\quad \mathcal{N}\overline{\mathcal{N}} = (\mathrm{p}\bar{\mathrm{p}} + \mathrm{n}\bar{\mathrm{n}})/2^{\frac{1}{2}} \qquad I = 0 \qquad I_3 = 0$

(8″) $\quad \mathcal{N}\overline{\mathcal{N}} = (\mathrm{n}\bar{\mathrm{n}} - \mathrm{p}\bar{\mathrm{p}})/2^{\frac{1}{2}} \qquad I = 1 \qquad I_3 = 0\,.$

The problem previously mentioned consists in the fact that the value of $J(\zeta)$ has been deduced by G.T. from the experimental data of the $\pi^{\pm} \to \mu^{\pm}\nu$ decay: in this case the intermediate states are

$$\mathcal{N}\overline{\mathcal{N}} = -\,\mathrm{n}\bar{\mathrm{p}} \quad \text{and} \quad -\,\mathrm{p} + \bar{\mathrm{n}}$$

The quasi coincidence of the decay probabilities of the two pionic processes (the corrections due to the final interactions being negligible in such case), indicates that the complex phase shifts for the $\mathrm{n}\bar{\mathrm{p}}$ and $\mathrm{p}\bar{\mathrm{n}}$ should be very similar.

The experimental data of the pion deay can be used for the $\Lambda \to \mathrm{p}\pi^-$ decay, while for he $\Lambda \to \mathrm{N}\pi^0$ decay there can be also a remarkable difference between the parameters $J(\zeta)$ evaluated with the three intermediate states (8), (8′) and (8″), if the intermediate complex phase shifts appearing in $J(\zeta)$ are sensibly different.

If one admits that in the strong $\mathcal{N}\overline{\mathcal{N}} \to \pi$ interaction the total isotopic spin is conserved, one has to take into account the intermediate states(8″) only: with regard to the Fermi interaction one has $\Delta I = \frac{1}{2}$ for (8′), $\Delta I = \frac{1}{2}$ or $\Delta I = \frac{3}{2}$ for (8) and (8″).

Unfortunately, reliable data for evaluating $J(\zeta)$ directly are not available at present; however, some indication can prove that the actual intermediate states are given by the (8″) and therefore that the isotopic spin is conserved in the $\mathcal{N}\overline{\mathcal{N}}$ annihilation for both the decays (6).

This fact is not important by itself, because one can justify the choise (8″) from the I-conservation in the $\mathcal{N}\overline{\mathcal{N}} \to \pi$ decay; the tentative evaluation of the phase shifts shows that presumably there is a small difference between the values of $J(-\omega_q^2)$ fot the states (8) and (8″)—and this is important for

(7) H. A. Bethe and J. Hamilton: *Nuovo Cimento*, **4**, 1 (1956).

the problem—while this difference seems to be stronger (by the same procedure) for the states (8′). Thus the states (8′) have been taken into account only to get a term of comparison.

The mentioned calculation (tentative in character) is based on the fact that the experimental $p\bar{p}$ scattering phase shifts are larger than the ones corresponding to the $n\bar{p}$ and $p\bar{n}$ scatterings: this is also intuitive when one takes into account the pionic and electromagnetic interactions in the two cases. Also the complex phase shifts for the $n\bar{n}$ interaction (in particular for the electromagnetic effects) differ from those for the $p\bar{p}$ interaction.

An approximate evaluation of the phase shifts of the different processes can be obtained by calculating the mean potential due to the pionic and electromagnetic interaction, amongst the different particles, and the matrix elements for the decay of the various pairs.

One can verify [8] that for the processes implying one or two pions, and by a Born approximation, at very low energies:

$$\varphi_{p\bar{p}} > \varphi_{n\bar{n}} > \varphi_{n\bar{p}}\,, \qquad \varphi_{p\bar{p}} : \varphi_{n\bar{n}} : \varphi_{n\bar{p}} \approx 1 : 0.7 : 0.6\,. \tag{9}$$

The most important contribution to $J(-\omega_q^2)$ comes out from the lowest values of ξ: the limitation (9) lead to a prevision of a value of $J(-\omega_q^2)$ for the $P\pi^-$ decay less than the value assumed by the same parameter for the $N\pi^0$ decay.

Unfortunately, the final results (as it is evident from the following Table I)

TABLE I. – (*Values of y (eq. 12)*).

x \ g	15	14.8	14.6
0.114	400	570	880
0.130	61	68	78
0.135	44	49	55

are rapidly variable functions of J, so that it is not satisfactory to draw quantitative conclusions from the (9).

The preceding discussion lead, on the other hand, to the conclusion that the sign of the ratio (10)—undefined if one refers to the pionic decay probability, which fixes its absolute value—is indeed positive.

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[8] The calculation was done with the standard method for evaluating the interaction amongst nucleons (*e.g.* in H. MIYAZAWA: *Phys. Rev.*, **104**, 1741 (1956)). For the decay processes we took into account the selection rules given by LEE and YANG (T. D. LEE and C. Y. YANG: *Nuovo Cimento*, **3**, 749 (1956)).

We can deduce, from the preceding considerations, that, once taken J from the relation of G.T.

$$x = \frac{J}{1 + G^2 J/2\pi^2} \simeq 0.13 \,, \tag{10}$$

one should expect for the axial part of the $\Lambda \to p\pi^-$ matrix element (proportional to x) a value, which is slightly different from that of the axial part of the $\Lambda \to n\pi^0$ matrix element, because J turns ont to be, in case, very small.

Then from the matrix element (2′) which, by eliminating the factors $\gamma_\Lambda(p_\Lambda - p_\pi)_\lambda$ and $\gamma_\lambda\gamma^5(p_\Lambda - p_\pi)_\lambda$, can be written in the form (indipendent from the isotopic spin of the final state):

$$\begin{cases} iF\tilde{u}_\pi(1 - \varrho\gamma_5)u_\Lambda \,, \\ F = \dfrac{M\sqrt{2}(M_\Lambda - M)}{2\pi^2} \dfrac{GJg_A}{1 + G^2J/2\pi^2} \,, \quad \varrho = \dfrac{M_\Lambda + M}{M_\Lambda - M} \dfrac{g_A}{g_V}\left(1 - \dfrac{G^2J}{2\pi^2}\right) \,, \end{cases} \tag{11}$$

one can deduce the value of the ratio between the probabilities of the $\Lambda \to p\pi^-$ and $\pi \to \mu\nu$ decays:

$$y = \frac{w(\Lambda \to p\pi^-)}{w(\pi \to \mu\nu)} \simeq 61 \,. \tag{12}$$

with

$$G^2/4 = 15 \,, \qquad g_V/g_A = -1.41/1.25 \,.$$

From the experimental values of the life-times:

$$\tau_{\pi\to\mu\nu} \simeq 2.60 \cdot 10^{-10} \text{ s} \,, \qquad \tau_{\Lambda\to p\pi} \simeq 2.56 \cdot 10^{-8} \text{ s} \,,$$

one gets a fairly good agreement between the G.T. theory of the pionic decay and the evaluation of the Λ-decay. A better agreement is obtained by using for the ratio (10) the more recent [9] value ($x \simeq 0.126$): this point will be discussed in the following.

We point out that the discussion by G.T. fixes essentially the ratio x of equation (10) and cannot give an exact value for J: in the present case the largest contribution comes from the vector part of the interaction, which is proportional to J. As observed by G.T., the value of the ratio (10) is practically indipendent from the value of the integral J. This confirms, in the

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[9] M. L. Goldberger and S. B. Treiman: *1958 Annual Internationational Conference at CERN* (cited [1]), p. 260.

present case, that a deviation in the derivation of the decay rate is a consequence of the explicit dependence of $w(\Lambda \to p\pi^-)$ on J, a fact which shall be discussed briefly in the following.

The constant ϱ and the asymmetry parameter $a = -2\varkappa(1+\varkappa^2)$, $\varkappa \simeq 0.052\,\varrho$ assume, with the preceding data, the values:

$$\varrho = 52.5\,, \qquad a = 0.65\,,$$

a is closed enough to the verified ([10]) minimum ($a \gtrsim 0.7$); also for ϱ and a the preceding consideration can be repeated.

With regard to the corrections of the used procedure, one can take into account the possibility that, in the Fermi interaction, the Λ-particle dissociates in a pair of spinors different from the precedings ($n\bar{n}$); *e.g.* in $\Lambda\bar{\Lambda}$ or in a pair of other hyperons.

This cannot give a large modification to the results: in fact, the interaction between the intermediate pair and the pion must be strong, in order to be comparable with the previous one, and then conserving the total isotopic spin and the strangeness: for such a reason a decay like *e.g.* $B\bar{\Lambda} \to \pi^0$ is not possible, and then the indicated processes cannot be simple, but have to occur through a series of intermediate states which reduce their importance.

Finally, as far as the final interactions *a*) and *b*) (already cited) are concerned, their contribution is immediately evaluated in the case of a time-reversal invariant interaction. In fact, as pointed out by several authors, the corrections due to the pion-nucleon scattering are negligible when the matrix element is evaluated in a frame of reference fixed with the Λ-particle.

With respect to the *b*) interaction, it is connected essentially with the problem whether the universal Fermi interactions refers to « pure » or « dressed » spinors. The problem has a certain importance, when one keeps in mind the constant reference—done in the preceding—to the pion decay, in which such kind of interaction are absent. One can also keeps in mind that we have used for J just that value deduced from the life-time of the pionic decay.

With respect to the dissociation of the Λ-particle into virtual states, these can be deduced, according to the scheme proposed ([11]) by Gell-Mann, by the lagrangian:

$$\mathscr{L}_{\Lambda\pi} = ig_{\Lambda\pi}[\bar{\Sigma}^0\pi^0\gamma_5\Lambda + \bar{\Sigma}^+\pi^-\gamma_5\Lambda + \bar{\Sigma}^-\pi^+\gamma_5\Lambda]\,, \tag{13}$$

$$\mathscr{L}_{\Lambda K} = ig_{\Lambda K}[\bar{P}K^-\gamma_5\Lambda + \bar{N}\pi^0\gamma_5\Lambda]\,. \tag{13'}$$

Amongst the hyperonic intermediate states, the $\Sigma^0\pi^0$ state cannot give

([10]) S. B. Treiman: cited ([1]); R. A. Dalitz: *Phys. Rev.*, **112**, 605 (1958).
([11]) M. Gell-Mann: *Phys. Rev.*, **106**, 1296 (1957).

any contribution to the nucleonic decay, becomes $\Sigma^0 \to \pi^0 + \gamma$. The other intermediate states, on the contrary, can give a direct contribution, because

$$(13'') \qquad \pi^+\Sigma^- \to \pi^+ N\pi^- \to P\pi^-, \qquad \pi^-\Sigma^+ \to \pi^- P\pi^0 \to N\pi^0 .$$

The dissociation of the Λ-particle according to (13″) gives rise to corrections which—by assuming the procedure by G.T. also for the Fermi interaction of the Σ (and there is no reason because this should not happen) with values almost equal for the parameters given above—appear as corrections at the vertex of the decay, according to the scheme:

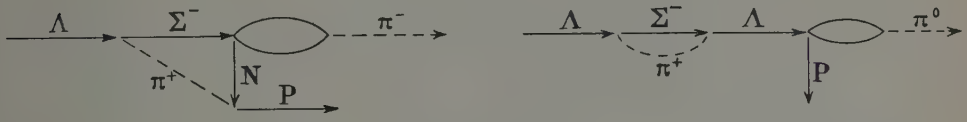

Fig. 1

and other similar pairs, for the processes (13). According to such graphs the final state in the matrix element (1) turns out to be an eigenstate incoming with respect to the pion-nucleon interaction; since the latter (like all the strong interactions) is time-reversal invariant, one realizes immediately—by the application of the Fermi-Watson theorem—that the dissociation $\Lambda \to \Sigma^{\pm}\pi^{\mp}$ does not alter appreciably the results and therefore the most important part of the corrections deriving from these processe is due to the probability that the Λ-particle is dissociated in a Σ^0 and a π^0 (in such a state it cannot give any contribution to the pionic decay).

Since it is experimental known that $g_{\Lambda\pi} \approx g_{N\pi}$ (constant for the pion-nucleon interaction), the cited contribution can be evaluated if one assumes the validity of the fixed source theory for the $\Lambda \cdot \pi$ interaction. In fact, it reduces to the evaluation of $\langle \tau_3 \rangle$.

With the preceding hypotheses, the probability to have the Λ-particle dissociated into $\Sigma^0\pi^0$ amounts to about 12%.

According to the Gell-Mann scheme (that is to the strangeness conservation) one can, on the other hand, exclude that an emission of K-mesons from the initial and their re-absorption by the final spinor field may occur, unless one wants to take into account complex processes, such as multiple interactions, whose contribution should be small. In fact it is experimentally known that $g_{\Lambda K} \ll g_{\Lambda\pi}$, and therefore the processes deriving from the lagrangian (13′) are negligible.

We must now point out that an even slight alteration of the ratio x of eq. (10) determines a strong variation on the decay state and the asymmetry parameter a.

One has here to remember that the largest influence appears in that part of the matrix element due to the vector Fermi interaction. We report in Table I a set of values for the ratio γ between the Λ- and pionic decay rates, as function of the ratio x and $g = G^2/4$.

The corrections due to the interactions from the graphs of Fig. 1, are the same both for $p\pi^-$ and $\mathcal{N}\pi^0$ decays ([12]). Because the strong dependence on J of the ratio x, even a slight difference between the cosine of complex phases for the intermediate states (8) and (8′) gives rise to a sensible difference amongst the parameters J for $\mathcal{N}\pi^0$ and $p\pi^-$ decays.

From the above cited calculation (equation (9)), such a difference in the parameters J cannot give us a satisfactory answer.

It can be observed that the asymmetry parameter a turns out a function which does not vary so rapidly as the ratio R: the knowledge of a for the $N\pi^0$ decay together with R could fix the value of J for such a process and, consequently, should be a good test for the Fermi theory of the hyperon decay interaction.

* * *

I'm glad to thank very much Professors M. Cini and R. Gatto for a useful conversation on the subject treated in the previous pages.

([12]) They correspond to $x = 0.114$ in Table I; for such a value, and $g = 15$ it is, $a = 0.33$ — a too small value.

Note added in proof.

It is now generally accepted that the rule $\Delta Q = \pm 1$ holds for each one of the two currents which gives rise to the Fermi interaction. Q is the charge of the particles whose operators appear in the currents. Applying this rule to the decay $\Lambda \to n\pi^0$ one avoids the difficulties previously mentioned and simplifies considerably the calculation. Indeed, in this case, the intermediate state of nucleon-antinucleon can be only $\bar{p}p$. Rearranging the nucleon lines at the Fermi vertex, there is therefore no a sensible difference between the two modes of pionic decay of the Λ-particle.

RIASSUNTO

Supponendo che il decadimento pionico della Λ sia da attribuire ad una interazione di Fermi, si calcolano — con il procedimento di Goldberger e Treiman — le correzioni alle costanti d'accoppiamento assiale e vettoriale. Il procedimento è direttamente comparabile a quello del decadimento pionico solo nel caso di decadimento $\Lambda \to p\pi^-$ (per il decadimento $\Lambda \to n\pi^0$ non si possono avere per ora dati sicuri). I risultati sono soddisfacenti.

IL NUOVO CIMENTO Vol. XIV, N. 2 16 Novembre 1959

Further Work on the Decay of Thallium 202.

S. Jha and H. G. Devare

Tata Institute of Fundamental Research - Bombay

(ricevuto il 6 Luglio 1959)

Summary. — Radiations from 12 day ^{202}Tl have been studied in scintillation coincidence spectrometers. In addition to the well-known γ-ray of 440 keV energy, a very low intensity γ-ray of energy 960 keV has been found showing that the electron capture decay of ^{202}Tl leads to the 440 keV and 960 keV states of ^{202}Hg. By the « sum-coincidence » technique and the conventional coincidence technique, it has been shown that the 960 keV state de-excites also by the cascade emission of 440 keV and 520 keV γ-rays. By measuring the ratio of the number of captures from the K shell and that from L, M ... shells, the decay energy of ^{202}Tl has been calculated to be (980^{-60}_{+760}) keV.

1. – Introducton.

In a recent paper (1), the ratio of the probability of the L electron captures P_L, and the K electron captures, P_K, in ^{202}Tl was measured by the summing technique in a single crystal and the decay energy was calculated (neglecting all captures of electrons from shells higher than the L shell) to be 600^{-5}_{-7} keV. Later, the ratio P_L/P_K for ^{202}Tl was measured also by Hamers *et al.* (2,3) and the decay energy was calculated to be (599 ± 4) keV. However, if one estimates the decay energy of ^{202}Tl from the β-decay systematics of Way and Wood (4), taking the decay energy of $^{204}\text{Tl} \rightarrow {}^{204}\text{Hg}$ to be 380 keV (5) and

(1) R. K. Gupta and S. Jha: *Nuovo Cimento*, **5**, 1524 (1957).
(2) P Kramer, H. C. Hamers and G. Meijer: *Physica*, **22**, 205 (1956).
(3) H. C. Hamers, A. Marseille and Th. J. DeBoer: *Physica*, **23**, 1056 (1957).
(4) K. Way and M. Wood: *Phys. Rev.*, **94**, 119 (1954).
(5) D. Strominger, J. M. Hollander and G. T. Seaborg: *Rev. Mod. Phys.*, **30**, 585 (1958).

that of $^{200}Tl \rightarrow {}^{200}H$ to be 2.46 MeV ([5]) and assuming the plot of the decay-energy against neutron number to be linear in this region also, one expects the decay energy of ^{202}Tl to be about 1.4 MeV ([5]). It is gathered that the workers in Amsterdam (*) have carefully studied the radiations from ^{202}Tl and have found that in addition to the 440 keV γ-ray, two more γ-rays of energy 520 keV and 960 keV are emitted in the decay of ^{202}Tl. This communication prompted us to examine the decay scheme of ^{202}Tl and to re-determine the decay energy of this isotope. The results of our study, which confirms the findings of the Dutch workers, and of a similar study in Stockholm ([6]), is being presented in this note.

2. Study of the high energy γ-rays.

The source of ^{202}Tl was obtained by the chemical separation from a mercury oxide target irradiated by a 25 MeV deuteron beam in the cyclotron of the Birmingham University. After the short-lived activities had died down very much, a search was made for the 960 keV γ-ray. A strong source of ^{202}Tl was used; the K X-rays and the 440 keV γ-ray were very much cut down by the use of 1.4 cm thick mercury absorber. The spectrum is reproduced in Fig. 1, where one can see a small peak at 960 keV. This peak persisted appearing even after repeated chemical purification. It was concluded that in the decay of ^{202}Tl, a γ-ray of

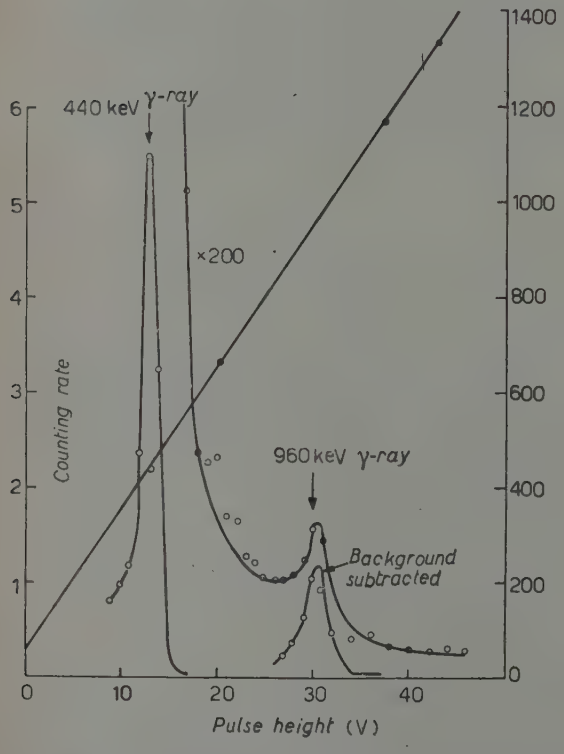

Fig. 1. – The γ-spectrum of ^{202}Tl with the source under 1.4 cm of mercury to absorb the 440 keV γ-ray. The 960 keV γ-ray peak is shown with an arrow.

(*) Private communication from Dr. HERRLANDER of the Nobel Institute of Physics, Stockholm.

([6]) R. K. GUPTA: Nobel Institute of Physics, Stockholm (private communication).

energy 960 keV was also emitted. Probably this γ-ray arose from some electron capture in ^{202}Tl leading to an excited state in ^{202}Hg at 960 keV. In such a case, one would expect a transition from the 960 keV state to the 440 keV state with the emission of a γ-ray of energy 520 keV.

3. – Coincidence studies.

The γ-rays were studied first by the « sum-coincidence » technique (7). The ^{202}Tl source was wrapped up in 2 mm thick Pb to cut down the K X-rays, The source was viewed by two NaI(Tl) crystal ($1\frac{1}{2}$ in. diameter and 2 in. high) and the photomultiplier (DuMont 6292) units. The voltage supplied to the dynodes of the two photomultipliers was so adjusted that for the same γ-ray falling on the two crystals, the output pulses were of the same height. With the ^{202}Tl source in-between, the output pulses from the two units were added

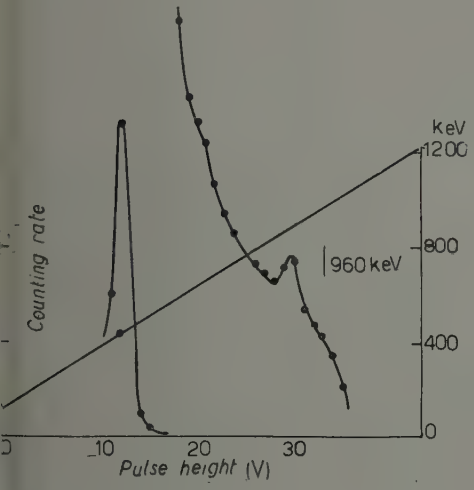

Fig. 2. – The sum spectrum of rays of ^{202}Tl with the source under 2 mm lead, showing the random sum peak of the 440 keV γ-ray at 880 keV and a small hump which is due to the small intensity 960 keV γ-ray and the summing of the 440 keV γ-ray and the possible 520 keV γ-ray.

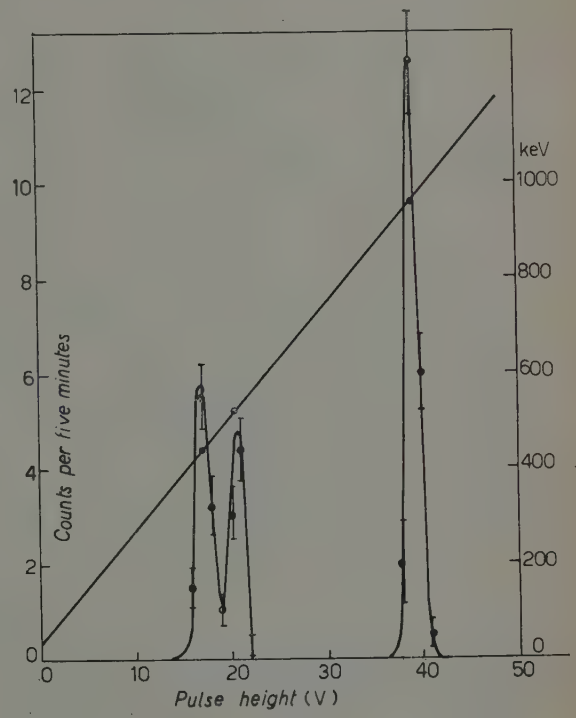

Fig. 3. – « Sum-coincidence » spectrum of ^{202}Tl, with the 960 keV γ-ray in the gate. Well-formed peaks at 440 keV and 520 keV show that the 960 keV state decays also by the cascade emission of 440 keV and 520 keV γ-rays.

(7) A. M. Hoogenboom: *Nucl. Instr.*, **3**, 57 (1958).

electronically and analysed in a single channel analyser. The added spectrum is given in Fig. 2. The peak at 880 keV is due to the random summing of the 440 keV γ-ray falling on the two crystals. The peak at 960 keV is partly due to the low intensity 960 keV γ-ray and partly to the summing of the 440 keV and the possible 520 keV γ-rays. A small portion (2 V) of the sum peak at 960 keV was taken in the gate of a slow-fast coincidence unit with the resolving time of $8 \cdot 10^{-8}$ s and the coincidence spectrum in one of the crystals was scanned. In such an arrangement, the photo-peaks of only those γ-rays, which add up to 960 keV, would appear. One can see from Fig. 3 that two peaks of almost equal height appear at 440 keV and 520 keV. From this it was concluded that ^{202}Tl decays also to a level at 960 keV in ^{202}Hg which de-excites itself by the emission of two γ-rays of 440 keV and 520 keV energy in cascade.

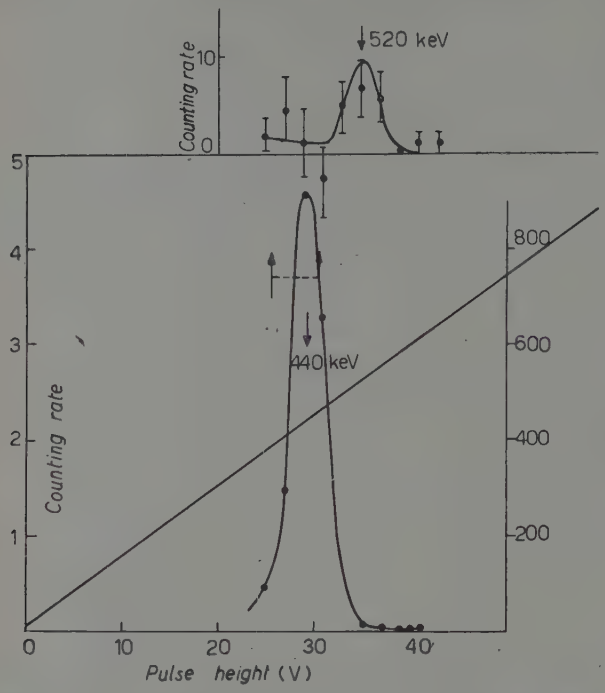

Fig. 4. – The coincidence spectrum of γ-rays of ^{202}Tl with the 440 keV γ-ray in the gate.

The detection of the 520 keV γ-ray in coincidence with the 440 keV γ-ray was tried also in the conventional coincidence spectrometer. In a slow-fast coincidence unit of resolving time $8 \cdot 10^{-8}$ s, the pulses of the 440 keV γ-ray were taken in the gate and the spectrum was scanned to detect the γ-rays in coincidence. Fig. 4 shows the coincidence spectrum, where the correction for the random coincidence has been made. One can see that, as expected, the 520 keV γ-ray peak again appears. This experiment confirms the conclusion of the « sum-coincidence » experiment.

4. – Decay energy measurement.

The previous measurement of P_L/P_K for ^{202}Tl had been made with a single crystal by the summing technique [1]. In the present studies the con-

ventional coincidence technique was adopted. The 440 keV γ-ray was received on a NaI(Tl) crystal of $1\frac{3}{4}$ in. diameter, 2 in. high, and the K X-ray was detected in a NaI(Tl) crystal of $1\frac{3}{4}$ in. diameter $\frac{1}{8}$ in. thickness which had an aluminium cover weighing about 5 mg/cm^2. The distance of the source from the thin crystal detector was kept about 10 cm to minimize the error in the calculation of the solid angle. From the counting rate of the 440 keV γ-ray in the gate and the coincidence counting rate of the K X-ray, the ratio $P_{L+M+N\ldots}/P_K$ was calculated from the relation

$$(N_{\gamma X}/N_\gamma) = (N_K/N)\{1/(\omega_K S\, e^{-\mu t})\}$$

and

$$\{1 - (N_K/N)\}/(N_K/N) = (P_{L+M+\ldots})/P_K\,,$$

where:

$N_{\gamma X}$ is number of K X-rays in coincidence with the 440 keV γ ray;

N_γ is number of 440 keV γ-rays in the gate;

N_K/N is ratio of the K electron captures to the total number of captures leading to the 440 keV state;

S is the solid angle subtended by the source at the X-ray detector;

ω_K is the K-fluorescence yield in $_{80}$Hg;

$e^{-\mu t}$ is the fraction of the mercury K X-ray transmitted through the thin aluminum cover the of X-ray detecting crystal;

$N_{L+M+N\ldots}$ is the probability of electron captures from the shells L, M, N....

From the experimental value of N_K/N, the ratio of the probability of the capture of electrons from the shells L, M, N, ... and the capture of electrons from the K shell can be calculated. From this ratio, the ratio P_L/P_K can be calculated from the relation ([8,9])

$$1.27\,(P_L/P_K) = P_{LMN\ldots}/P_K\,.$$

From the value of P_L/P_K thus found, the decay energy of ^{202}Tl to the 440 keV state has been calculated from the relation ([10])

$$P_L/P_K = (g_{L_I}^2/g_K^2)\{1 + (f_{L_{II}}^2/g_K^2)\}\{(W - B_L)^2/(W - B_K)^2\}\,.$$

([8]) G. J. Nijgh, A. H. Wapstra, L. Th. M. Ornstein N. Salomons-Grobben and J. R. Huizenga: *Nucl. Phys.*, **9**, 528 (1958-59).

([9]) A. H. Wapstra, G. J. Nijgh and R. Van Lieshout: *Nuclear Spectroscopy Tables* (1959).

([10]) H. Brysk and M. E. Rose: *Rev. Mod. Phys.*, **30**, 1169 (1958).

where: $g^2_{L_I}/g^2_K$ is the ratio of the the probabilities of finding the L_I and K electrons at the nucleus;

$f^2_{L_{II}}/g^2_K$ is the ratio of probabilities of finding the L_{II} and K electrons at the nucleus;

W is the decay energy in keV;

B_K and B_L are the K electron and L electron binding energies.

The results are given below in Table I.

TABLE I.

N_γ	$N_{\gamma X}$	S	ω_K	$e^{-\mu t}$	$\frac{P_{L+M+\ldots}}{P_K}$	P_L/P_K	P_{L_I}/P_K	Decay energy to 440 keV state	Total decay energy
$17\,757 \pm 296$	$1\,521 \pm 18$	0.011 5	0.946 ± 0.008	0.99	$0.244^{+0.027}_{-0.024}$	$0.192^{+0.022}_{-0.019}$	$0.19^{+0.022}_{-0.019}$	540^{-60}_{+760} keV	980^{-60}_{+760} keV

The decay energy of ^{202}Tl has been measured very carefully at the Nobel Institute to be 1150^{+50}_{-50} keV ([6]).

In the course of the measurements, it was found that the decay energy was rather large; and so in order to get the decay energy within narrow limits, one needed to measure the ratio $P_{L+M+\ldots}/P_K$, the solid angle, and the absorption of the K X-ray in the cover of the crystal with an accuracy better than 1% and to know the value of the K fluorescence yield with the same accuracy. It has been possible to see that the earlier measurements gave too low a value of the decay energy because the electron capture from the higher shells was neglected, and the value of the solid angle calculated from the thickness of the cover of the NaI crystal was not accurate enough. If the decay energy is to be calculated from the measurements by the summing technique one must know the solid angle rather accurately.

5. – Discussion.

From the results of these measurements, and those from Stockholm and Amsterdam, it appears definitely that the electron capture decay of ^{202}Tl leads predominantly to the 440 keV state of ^{202}Hg, but there is a small branching to the 960 keV state also. The intensity of the electron capture branching to the 960 keV state has been estimated from the intensity of the 960 keV γ-ray and that of the 520 keV γ-ray. The intensity of the 520 keV γ-ray, which is not resolved in the single spectrum on a scintillation spectrometer, has been estimated from the ratio of the coincidence count for the full 520 keV peak and the 440 keV gate count. The sum of the intensity of the 960 keV γ-ray

and the 520 keV γ-ray is about 0.3% of the intensity of the 440 keV γ-ray. The electron capture branching to the 960 keV state thus comes out to be 0.3% of the intensity of captures leading to the 440 keV state. If one assumes that the transition to the ground state is, contrary to the results of KRAMER *et al.* (2), negligible, then about 0.3% of the electron capture transition leads to the 960 keV state and about 99.7% transition leads to the 440 keV state. This fact, in conjunction with the decay energy data gives the log *ft* values 8.8 and 7.3 respectively for the transitions to the second and the first excited states. The log *ft* value 8.8 is rather large for the first-forbidden transition of the type $\Delta J = 0, \pm 1$, yes. From the measured values of spin and parity 2^- for the ground state of ^{202}Tl and the expected spin and parity values of 2^+ and 2^+ for the first and the second excited states of the even-even nuclide ^{202}Hg, one would not normally expect the log *ft* values to be larger than 7.5.

* * *

Our thanks are due to Mr. R. M. SINGRU, Mr. K. S. BHATKI and Mr. K. P. GOPINATHAN for t e help we have received from them. We are grateful to Dr. HERRLANDER of the Nobel Institute of Physics, Stockholm, for communicating to us the results of the Amsterdam workers and to Mr. R. K. GUPTA for informing us about his results before publication.

RIASSUNTO (*)

Sono state studiate in spettrometri a scintillazioni in coincidenza le radiazioni da parte del ^{202}Tl di 12 giorni. Oltre ai ben noti raggi γ di 440 keV di energia, si sono trovati raggi γ di 960 keV di intensità molto debole, la qualcosa dimostra che il decadimento per cattura elettronica conduce agli stati energetici del ^{202}Hg di 440 keV e 960 keV. Facendo uso della tecnica della « somma delle coincidenze » e della ordinaria tecnica delle coincidenze, si è provato che lo stato 960 keV si diseccita anche con l'emissione a cascata di raggi γ di 440 keV e di 520 keV. Calcolando il rapporto fra il numero di catture rispettivamente da parte dello strato K e degli strati L, M, ..., si ha che l'energia di decadimento del ^{202}Tl è uguale a (980^{-60}_{+760}) keV.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIV, N. 3 1° Novembre 1959

A Mandelstam Representation in Potential Scattering.

J. Bowcock and A. Martin

CERN - Geneva

(ricevuto il 7 Luglio 1959)

Summary. — The scattering from a given class of potentials including exchange forces is considered. For fixed energy one may prove the scattering amplitude is analytic in the complex $\cos\theta$ plane inside the Lehmann ellipse and that each term of the Born series has the cuts and analytic behaviour conjectured by Mandelstam for the field theoretical case. A Mandelstam representation is written for each term of the Born series.

Introduction.

It has been suggested recently that the S-matrix for scattering processes may be derived from conjectured analytic properties of the T-matrix, that is analytic properties in the complex planes obtained by letting various physical quantities, such as energy, become complex (1). On the other hand, Charap and Fubini (2) have shown that for nucleon-nucleon scattering in the energy range < 200 MeV, one may define a potential which gives the same T-matrix as field theory to each order in perturbation theory. It is therefore extremely interesting to investigate the analytic properties of the scattering amplitude obtained from the Schrödinger equation with certain potentials.

The analytic properties of the Schrödinger scattering amplitude have been considered previously by Khuri (3) and by Bowcock and Walecka (4) (hereafter denoted by I). Khuri investigated analytic properties in the complex energy plane keeping the momentum transfer fixed, whereas the latter authors

(1) S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).
(2) J. Charap and S. Fubini: to be published in *Nuovo Cimento*.
(3) N. N. Khuri: *Phys. Rev.*, **107**, 1148 (1957).
(4) J. Bowcock and D. Walecka: preprint to be published in *Nucl. Phys.*

considered the behaviour in the complex energy plane keeping the scattering angle fixed.

LEHMANN has shown that for the field theoretical case, the scattering amplitude is analytic in the $\cos\theta$ plane ([5]) (keeping the energy fixed and positive) inside a given ellipse. Here we wish to prove the same statement for the case of scattering from a potential which may be represented as a superposition of Yukawa potentials including an exchange force (*). It is also shown that each term of the Born series has analytic properties in the $\cos\theta$ plane which are identical with those of the Mandelstam conjecture. In fact, we show that by using the analytic properties in both the energy and $\cos\theta$ planes, one may write a Mandelstam representation for each f_n (**).

1. – Analytic properties of each term of the Born series.

First let us for simplicity consider the scattering from a simple Yukawa potential $(\lambda/M)(\exp[-\mu r]/r)$. Then the scattering amplitude $f(k^2, \cos\theta)$ may be written as (see I):

$$f(k^2, \cos\theta) = -\frac{\lambda}{4\pi}\left[\frac{4\pi}{\mu^2 + (\boldsymbol{k}' - \boldsymbol{k})^2} + \sum_{n=1}^{\infty} \lambda^n f_n\right], \tag{1.1}$$

where as an integral over configuration spaces

$$f_n \equiv f_n^c = \left(-\frac{1}{4\pi}\right)^n \int \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{z}\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_{n-1} \exp\,[-i\boldsymbol{k}'\cdot\boldsymbol{y}]\, \frac{\exp[-\mu y]}{y} \cdot \frac{\exp[ik|\boldsymbol{y} - \boldsymbol{x}_1|]}{\boldsymbol{y} - \boldsymbol{x}_1} \ldots \frac{\exp[ik|\boldsymbol{x}_{n-1} - \boldsymbol{z}|]}{|\boldsymbol{x}_{n-1} - z|}\, \frac{\exp[-\mu z]}{z} \exp[i\boldsymbol{k}\cdot\boldsymbol{z}]. \tag{1.2}$$

or as an integral over momentum spaces

$$f_n \equiv f_n^K = \operatorname*{Lim}_{\varepsilon \to 0} 4\pi \left(\frac{-1}{2\pi^2}\right)^n \cdot \int \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_n \frac{1}{\mu^2 + (\boldsymbol{k}' - \boldsymbol{k}_1)^2} \frac{1}{(k_1^2 - k^2 - i\varepsilon)} \frac{1}{\mu^2 + (\boldsymbol{k}_1 - \boldsymbol{k}_2)^2} \cdots \frac{1}{\mu^2 + (\boldsymbol{k}_n - \boldsymbol{k})^2}. \tag{1.3}$$

In order to put f_n into a form which readily demonstrates its analytic pro-

([5]) H. LEHMANN: *Nuovo Cimento*, **10**, 578 (1958).

(*) This has been proven also by R. JOST: private communication from H. LEHMANN.

(**) Professor G. F. CHEW informed us that similar work was being done by M. L. GOLDBERGER and N. KHURI.

perties in the complex $\cos\theta$ plane, we make use of the fact that for three unit vectors $\boldsymbol{u}'$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and for α_1, α_2 real and greater than 1

$$\int \frac{\mathrm{d}\Omega'}{[\alpha_1 - \boldsymbol{u}_1\cdot\boldsymbol{u}'][\alpha_2 - \boldsymbol{u}'\cdot\boldsymbol{u}_2]} = -2\pi\cdot \int_{-1}^{1} \frac{\mathrm{d}y}{\left\{\alpha_1\frac{(1+y)}{2} + \alpha_2\frac{(1-y)}{2}\right\}^2 - \frac{(1+y^2)}{2} - \frac{(1-y^2)}{2}\,\boldsymbol{u}_1\cdot\boldsymbol{u}_2} \tag{1.4}$$

This equality may be proved by using the Feynman parametrization

$$\frac{1}{ab} = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}y}{\left\{a\frac{(1+y)}{2} + b\frac{(1-y)}{2}\right\}^2} \qquad (a \text{ and } b \text{ both positive})$$

for the integrand. This enables one to do the integration over Ω' resulting in (1.4).

If we group together all the terms in the integrand of (1.3) which have the form $1/(\mu^2+(\boldsymbol{k}_i-\boldsymbol{k}_j)^2)$, we may write these as

$$\frac{1}{\mu^2-(\boldsymbol{k}'-\boldsymbol{k}_1)^2}\left[\prod_{i=1}^{n}\frac{1}{\mu^2+(\boldsymbol{k}_i-\boldsymbol{k}_{i+1})^2}\right]\frac{1}{\mu^2+(\boldsymbol{k}_n-\boldsymbol{k})^2} \equiv \frac{1}{2kk_1\left(\frac{\mu^2+k^2+k_1^2}{2kk_1}\right)-\boldsymbol{u}'\cdot\boldsymbol{u}_1} \cdot \left[\prod_{i=1}^{n}\frac{1}{2k_ik_{i+1}\left\{\frac{\mu^2+k_i^2+k_{i+1}^2}{2k_ik_{i+1}}-\boldsymbol{u}_j\cdot\boldsymbol{u}_{j+1}\right\}}\right]\frac{1}{2k_nk\left\{\frac{\mu^2+k_n^2+k^2}{2kk_n}-\boldsymbol{u}_n\cdot\boldsymbol{u}\right\}}$$

Since the angular variables $\Omega_1, \ldots, \Omega_n$ belonging to $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n$ do not occur in the Green's functions, we may write

$$\int \frac{\mathrm{d}\Omega_1}{\left(\frac{\mu^2+k^2+k_1^2}{2kk_1}-\boldsymbol{u}'\cdot\boldsymbol{u}_1\right)\left(\frac{\mu^2+k_1^2+k_2^2}{2k_1k_2}-\boldsymbol{u}_1\cdot\boldsymbol{u}_2\right)} = 2\pi\int_{-1}^{+1}\frac{\mathrm{d}y_1}{[\alpha_1'(k,k_1,k_2,y_1)-\boldsymbol{u}_1\cdot\boldsymbol{u}_2]\frac{(1-y_1^2)}{2}} \tag{1.5}$$

where α_1' is given given by Eq. (1.4).

Continuing in this way, one may do the integration over Ω_2, Ω_3, ... arriving finally at

$$(1.6) \qquad f_n^P = \lim_{\varepsilon \to 0} 4\pi \left(\frac{1}{\pi}\right)^n \cdot \int \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_n \ \mathrm{d}k_1 \ldots \mathrm{d}k_n}{2k^2[\beta_n(k, k_1 \ldots k_n; y_1 \ldots y_n) - \cos\theta] \prod_{i=1}^{n} (1 - y_i^2)(k_i^2 - k^2 - i\varepsilon)} .$$

This form of f_n is very convenient for making an analytic continuation from the physical values of $\cos\theta$, *i.e.* $-1 \leqslant \cos\theta \leqslant 1$ into the whole complex $\cos\theta$ plane (*). $\beta(k, k_1 \ldots k_n; y_1 \ldots y_n)$ is a real function and one may show (Appendix I) that its minimum value as a function of all the variables is $1 + ((n+1)^2\mu^2/2k^2)$ and its maximum $+\infty$. At the border of the domain of the auxiliary variables, the integrand may be singular, but, however, it can be proven that the integral (1.6) exists.

As is well known, an integral of this form having a vanishing denominator for all values of $\cos\theta \geqslant 1 + ((n+1)^2\mu^2/2k^2)$ represents a function which is analytic in the whole complex $\cos\theta$ plane, but having a cut along the positive real axis from $1 + ((n+1)^2\mu^2/2k^2)$ to infinity. One may also demonstrate (Appendix II) that f_n^P goes to zero on a large circle with centre the origin as the radius of the circle goes to infinity.

Let us now consider the case when there is an exchange force present, *i.e.* when we may write

$$V(r) = \lambda_1 \frac{\exp[-\mu r]}{r} + \lambda_2 \frac{\exp[-\mu r]}{r} P_M ,$$

where P_M is the Majorana exchange operator. Using the same technique, one may easily show that the same analytic properties of f_n^P still hold with the exception of an extra cut which appears on the negative real axis running from $-(1 + (n+1)^2\mu^2/2k^2)$ to $\to \infty$.

It is simple to demonstrate that these analytic properties are also possessed by a more general potential which may be written in the form

$$V(r) = \int_{\mu}^{\infty} \left\{ \varrho_D(\sigma) \frac{\exp[-\sigma r]}{r} + \varrho_E(\sigma) \frac{\exp[-\sigma r]}{r} P_M \right\} \mathrm{d}\sigma$$

(*) At the same analyticity of the function « Im f_n^P » can be established in the same domain, because of the reality of f_n for k^2 real, positive.

2. – Mandelstam representation for f.

We are concerned in this Section to show that f_n may be written

$$f_n(s,t,u) = \iint \frac{\varrho_1(s',t')\,\mathrm{d}s'\,\mathrm{d}t'}{(s'-s)(t'-t)} + \iint \frac{\varrho_2(s',u')\,\mathrm{d}s'\,\mathrm{d}u'}{(s'-s)(u'-u)}\,, \tag{2.1}$$

where $s = 4k^2$,

$t = -2k^2(1-\cos\theta)$,

$u = -2k^2(1+\cos\theta)$,

and therefore $s+t+u=0$.

To do this, we use first the analytic properties of $f_n(s,t,u)$ in the s-plane with fixed t or with fixed u and also the analytic properties of f in the $\cos\theta$ plane proved in the last section.

KHURI has shown that the total scattering amplitude f for fixed t is analytic and goes to zero in all directions in the s-plane with the exception of poles corresponding to bound states of the potential, and also with a cut along the positive real axis extending from the origin to infinity. However, this proof is only for potentials with no exchange force. HAMILTON [6] has shown that for a potential including exchange force, one must split f into two parts $f^{(0)}+f^{(e)}$. $f^{(0)}$ corresponds to those terms in which the exchange potential acts an odd number of times, $f^{(e)}$ when it acts an even number of times. Then one may write a dispersion relation for each $f^{(i)}$ (or f_n) separately. For the case with no bound states

$$\begin{cases} f_n^{(e)}(s,t) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\operatorname{Im} f_n^{(e)}(s',t)\,\mathrm{d}s'}{s'-s-i\varepsilon}\,, \\ f_n^{(0)}(s,u) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\operatorname{Im} f_n^{(0)}(s',u)\,\mathrm{d}s'}{s'-s-i\varepsilon}\,, \end{cases} \tag{2.2}$$

$$f_n(s,t,u) = \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} f_n^{(0)}(s',u)\,\mathrm{d}s'}{s'-s} + \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} f_n^{(e)}(s',t)\,\mathrm{d}s'}{s'-s}\,. \tag{2.3}$$

We now wish to use the analytic properties of f_n we demonstrated in the $\cos\theta$ plane to rewrite $\operatorname{Im} f_n^{(0)}$ and $\operatorname{Im} f_n^{(e)}$ in terms of integrals over u' and t'.

---

[6] J. HAMILTON: preprint.

Although we proved the properties not for $f_n^{(e)}$ and $f_n^{(0)}$ separately, but just for f_n it is easy to see that the properties are valid for each separately, merely by setting either the direct or exchange part of the potential equal to zero in the general case. A simple transformation, namely $\cos\theta = 1+(t/2k^2)$ gives us the analytic properties in the t plane from those in the $\cos\theta$ plane. The only change is that for the direct term, the cut now runs from $(n+1)^2\mu^2$ to ∞ in the t plane. Similarly, one finds for $f^{(0)}$ in the u plane a cut running from $(n+1)^2\mu^2$ to ∞.

Thus we have

$$(2.4)\qquad \begin{cases} \operatorname{Im} f_n^{(0)}(s', u) = \dfrac{1}{\pi}\displaystyle\int_{(n+1)^2\mu^2}^{\infty} \frac{\varrho_n^{(0)}(s', u')\,\mathrm{d}u'}{u'-u}\,, \\ \operatorname{Im} f_n^{(e)}(s', t) = \dfrac{1}{\pi}\displaystyle\int_{(n+1)^2\mu^2}^{\infty} \frac{\varrho_n^{(e)}(s', t')\,\mathrm{d}t'}{t'-t}\,. \end{cases}$$

Thus

$$(2.5)\quad f_n(s, t. u) = \left(\frac{1}{\pi}\right)^2 \int_0^{\infty} \mathrm{d}s' \int_{(n+1)^2\mu^2}^{\infty} \mathrm{d}u' \, \frac{\varrho_n^{(0)}(s', u')}{(s'-s)(u'-u)} + \left(\frac{1}{\pi}\right)^2 \int_0^{\infty} \mathrm{d}s' \int_{(n+1)^2\mu^2}^{\infty} \mathrm{d}t' \, \frac{\varrho_n^{(e)}(s', t')}{(s'-s)(t'-t)}\,.$$

This is just of the form (2.1).

If we assume that the whole Born series is convergent and that summation and integration can be exchanged, we get for the total scattering amplitude, the following structure, when one considers a superposition of Yukawa potentials:

$$f = \underbrace{\frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{\varrho_B(u')\,\mathrm{d}u'}{u'-u} + \frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{\varrho_B(t')\,\mathrm{d}t'}{t'-t}}_{\text{Born Terms}} + \frac{1}{\pi^2}\int_0^{\infty}\mathrm{d}s'\int_{(2\mu)^2}^{\infty}\mathrm{d}u'\,\frac{\varrho_1(s', u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int_0^{\infty}\mathrm{d}s'\int_{(2\mu)^2}^{\infty}\mathrm{d}t'\,\frac{\varrho_2(s', t')}{(s'-s)(t'-t)}\,.$$

Let us now compare this result with the representation assumed in field theory by Mandelstam for nucleon-nucleon scattering (for this comparison, one should keep in mind that the quantity s defined here differs by $4M^2$ from the one used in the relativistic case). In the field theoretical case, a third weight function is present, giving to the scattering amplitude a contribution of the form

$$\iint \frac{\varrho_3(u', t')}{(u'-u)(t'-t)}\,\mathrm{d}u'\,\mathrm{d}t'\,.$$

One has to understand why, in the case of non-relativistic potential scattering, this term is *apparently* missing. One should keep in mind that in non-relativistic problems, u and t are of the order of $2k^2 \ll (2M)^2$. Now it has been noticed by Professor FUBINI, that the domain where the weight function differs from zero is made of two parts:

$$\text{either} \quad (2\mu)^2 < u' < \infty\,, \quad (2M)^2 < t' < \infty\,,$$
$$\text{or} \quad (2M)^2 < u' < \infty\,, \quad (2\mu)^2 < t' < \infty\,.$$

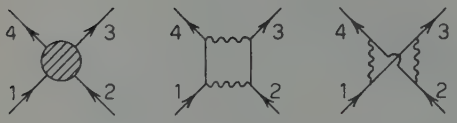

Fig. 1.

This can be understood in the following way (see Fig. 1): when particles 1 3 (related to t) can make a system of two or more pions, particles 1 4 (related to u) cannot make a system with mass less than $2M$. This is illustrated by fourth order graphs. When (u', t') lies in the first part of the domain, the dependence of the integral on t is negligible, and vice versa. So

$$\iint \frac{\varrho_3(u', t')\,\mathrm{d}t'\,\mathrm{d}u'}{(u'-u)(t'-t)} \simeq \int_{(2\mu)^2}^{\infty} \frac{\varrho_3'(t')\,\mathrm{d}t'}{t'-t} + \int_{(2\mu)^2}^{\infty} \frac{\varrho_3''(u')\,\mathrm{d}u'}{u'-u}\,.$$

In the case of potential scattering, such terms are not really missing, but they are included in the Born terms. This remark is in agreement with the conclusion of CHARAP and FUBINI, that it is possible to reproduce with a potential the main features of a field theoretical scattering matrix, if one restricts oneself to kinetic energies below the meson threshold.

3. – Analytic properties of the whole scattering amplitude.

In order to prove analytic properties of the total scattering amplitude, one has to prove the convergence of the Born Series. The only region in which we have been able to do this is inside the Lehmann ellipse.

To carry through the proof, we look at f_n in the co-ordinate representation eq. (1.2) Introduce unit orthogonal vectors $\hat{n}_-$ and $\hat{n}_+$ in the directions in the directions $\boldsymbol{k}' - \boldsymbol{k}$ and $\boldsymbol{k}' + \boldsymbol{k}$ respectively; f_n^o becomes

$$(3.1)\quad f_n^o = \left(-\frac{1}{4\pi}\right)^n \int \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_{n-1} \cdot$$
$$\cdot \exp\left[-i\sqrt{\frac{k^2}{2}(1-\cos\theta)}\,\hat{n}_- \cdot (\boldsymbol{y}+\boldsymbol{z})\right] \frac{\exp[-\mu y]}{y} \ldots \frac{\exp[-\mu z]}{y} \cdot$$
$$\cdot \exp\left[-i\sqrt{\frac{k^2}{2}(1+\cos\theta)}\,\hat{n}_+ \cdot + \cdot (\boldsymbol{y}-\boldsymbol{z})\right].$$

To define f_n^c for complex $\cos\theta$ we keep the unit vectors real. Then we are free to take them as co-ordinal axis. Considering the integration over $\boldsymbol{y}$ and $\boldsymbol{z}$ in cartesian co-ordinates (components y_1, y_2, y_3, z_1, z_2, z_3) the condition for the convergence of the integration over $\boldsymbol{y}$ is

$$\text{(3.2)} \qquad \operatorname{Im}\sqrt{1-\cos\theta}\, y_1 + \operatorname{Im}\sqrt{1+\cos\theta}\, y_2 \leqslant \frac{\mu}{k}\sqrt{2}\, y$$

or

$$\text{(3.3)} \qquad |\operatorname{Im}\sqrt{1-\cos\theta}|^2 + |\operatorname{Im}\sqrt{1+\cos\theta}|^2 \leqslant \left(\frac{\mu}{k}\sqrt{2}\right)^2$$

(3.3) is also the condition for convergence of the integration over $\boldsymbol{z}$. It is easy to show that this condition restricts $\cos\theta$ to lie inside an ellipse in the complex $\cos\theta$ plane with foci at -1 and $+1$ and large semi-axis $1+(2\mu^2/k^2)$.

One may show that the integration over the other variables $\boldsymbol{x}_1 \ldots \boldsymbol{x}_{n-1}$ is not divergent and in fact places an upper bound on f_n^c. We have

$$|f_n^c| \leqslant \left(\frac{1}{4\pi}\right)^n \int \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{z}\, \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_{n-1} \left| \exp\left[-i\sqrt{\frac{k^2}{2}(1-\cos\theta)}\, \hat{n}_- \cdot (\boldsymbol{y}+\boldsymbol{z})\right]\right| \cdot$$
$$\cdot \frac{\exp[-\mu y]}{y} \frac{1}{|\boldsymbol{y}-\boldsymbol{x}_1|} \frac{\exp[-\mu x_1]}{x_1} \cdots \frac{1}{|\boldsymbol{x}_{n-1}-\boldsymbol{z}|} \Bigg| \cdot$$
$$\cdot \exp\left[-i\sqrt{\frac{k^2}{2}(1+\cos\theta)}\, \hat{n}_+ \cdot (\boldsymbol{y}-\boldsymbol{z})\right] \Bigg| .$$

It was shown in (I) Appendix II, that

$$\int_0^\infty \frac{\mathrm{d}\boldsymbol{z}}{|\boldsymbol{x}-\boldsymbol{z}|} f(z) \leqslant \int_0^\infty z\, \mathrm{d}z\, |f(z)| .$$

Defining

$$\int_0^\infty y \frac{\exp[-\mu y]}{y} \mathrm{d}y = \frac{1}{\mu} = M$$

and

$$\int y\, \mathrm{d}y \frac{\exp[-\mu y]}{y} \left| \exp\left[-i\left(\sqrt{\frac{k^2}{2}(1-\cos\theta)}\, \hat{n}_- + \sqrt{\frac{k^2}{2}(1+\cos\theta)} \cdot \hat{n}_+\right) \cdot \boldsymbol{y}\right] \right| = N$$

we have

$$|f_n^c| \leqslant 4\pi N^2 M^{n-1} ,$$

$$\sum_n \lambda_n |f_n^c| \leqslant 4\pi \frac{N^2}{1-\lambda M}$$

Thus the Born series converges for $\lambda M \leqslant 1$ or $\lambda \leqslant 1/\mu$. Since each term of the series is analytic inside the Lehmann ellipse, we have thus shown that the total scattering amplitude f is also analytic inside that region.

* * *

We wish to thank Professor S. FUBINI for his continued interest and many fruitful discussions during the course of this work.

Note added in proof.

I was pointed out to us by Dr. OMNÈS and Dr. FROISSART that the derivation of equation (2.1) was not completely rigorous because it was not established that the order of integration over the two variables may be changed. However, it can be shown in a rigorous way that the knowledge of the analytic properties in $\cos\theta$ for real energy, when combined with Khuri's results, is sufficient to establish the analyticity in the two variables (T. REGGE, preprint).

APPENDIX I

What we wish to show is that $\min \beta_n(k, k_1 \ldots k_n, y_1 \ldots y_n) = 1 + ((n+1)^2\mu^2/2k^2))$. To do this we establish that if in eq. (1.4) the r.h.s. is written

$$-2\pi \int_{-1}^{1} \frac{\mathrm{d}y}{(a_1' - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2)((1-y^2)/2)}$$

and

$$\alpha_1 = \frac{N^2\mu^2 + k^2 + k_i^2}{2kk_i}; \qquad \alpha_2 = \frac{\mu^2 + k_i^2 + k_{i+1}^2}{2k_ik_{i+1}}.$$

Then

$$\min a_1' = \frac{(N+1)^2\mu^2 + k^2 + k_{i+1}^2}{2kk_{i+1}}.$$

Now

$$a_1' = \frac{2\left[\frac{N\mu^2 + k^2 + k_i^2}{4kk_i}(1+y) + \frac{\mu^2 + k_i^2 + k_{i+1}^2}{4k_ik_{i+1}}(1-y)\right]^2}{1-y^2} - \frac{1+y^2}{1-y^2}.$$

By simple differentiation, it is easy to minimize a_1' as a function of k_i giving

$$a_1' \geqslant \frac{\frac{1}{2}\left\{\frac{N^2\mu^2 + k^2}{k}(1+y) + \frac{\mu^2 + k_{i+1}^2}{k_{i+1}}(1-y)\right\}\left\{\frac{1+y}{k} + \frac{1-y}{k_3}\right\}}{1-y^2} - \frac{1+y^2}{1-y^2}.$$

Letting $t = (1+y)/(1-y)$, this becomes

$$a_1' \geqslant \frac{1}{2}\left\{\frac{N^2\mu^2 + k^2}{k}\, t + \frac{\mu^2 + k_{i+1}^2}{k_{i+1}}\right\}\left\{\frac{1}{k} + \frac{1}{t}\,\frac{1}{k_{i+1}}\right\} - \frac{1}{2}\left(t + \frac{1}{t}\right).$$

Minimizing with respect to t gives

$$a_1' \geqslant \frac{(N+1)^2\mu^2 + k^2 + k_{i+1}^2}{2kk_{i+1}}.$$

Now apply this to the first terms of f_n. We find

$$\alpha_1' \geqslant \frac{4\mu^2 + k^2 + k_2^2}{2kk_2}.$$

After the integration over the next angular variable Ω_2 we have

$$\alpha_2' \geqslant \frac{9\mu^2 + k^2 + k_3^2}{2kk_3}.$$

Repeating this process, we find

$$\beta_n \geqslant \frac{(n+1)^2\mu^2 + 2k^2}{2k^2} = 1 + \frac{(n+1)^2\mu^2}{2k^2}$$

Appendix II

We wish here to show that $|f_n| \to 0$ as $|\cos\theta| \to \infty$ in any direction other than the positive real axis.

Consider

$$f_n^P = \int \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_n}{(\beta_n - \cos\theta)} \prod_i \frac{\mathrm{d}k_i}{(1-y_i^2)(k_i^2 - k^2 - i\varepsilon)}.$$

$$\frac{1}{k_i^2 - k^2 - i\varepsilon} \quad \text{may be replaced by} \quad \frac{P_i}{k_i^2 - k^2} + i\pi\,\delta(k_i^2 - k^2),$$

where P_i denotes the principal value. In this way f_n may be expressed as a sum of terms, each of which has a certain number of principal value operators and a certain number of δ-functions.

Let us consider the term in which one has all principal value integrations and no δ-function, since the other terms may be treated in an analogous manner. This term is

$$\text{(A.1)} \qquad P_1 P_2 \ldots P_n \int \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_n}{\beta_n \left[{}^{k\,k_1 \ldots k_n}_{y_1 \ldots y_n}\right] - \cos\theta} \prod_i \frac{\mathrm{d}k_i}{(1-y_i^2)(k_i^2 - k^2)}.$$

Using the fact that

$$P\int_0^\infty \frac{f(x)\,\mathrm{d}x}{x^2-a^2} = \int_0^\infty \frac{f(x)-f(a)}{x^2-a^2}\,\mathrm{d}x \tag{A.2}$$

and suppressing the explicit dependence of β_n on the y's

$$P_2P_3\ldots P_n\int \mathrm{d}y_1\ldots\mathrm{d}y_n \left\{\frac{\beta_n[k,k_1,k_2,\ldots k_n]-\beta_n[k_i\,k_1,k_2\ldots k_n]}{(\beta_n[k,k_1,k_2\ldots k_n]-\cos\theta)(\beta_n[k,k_1,k_2\ldots k_n]-\cos\theta)}\right\}\cdot \prod_i \frac{\mathrm{d}k_i^2}{(1-y_i^2)(k_i^2-k^2)}\,. \tag{A.3}$$

Continuing using (A.2) for the integration over k_2, k_3, etc., we find that (A.1) reduces to

$$\int \mathrm{d}y_1\ldots\mathrm{d}y_n \frac{\sum_{i=1}^{i=2^n-2} b_i\cos^i\theta}{\sum_{i-1}^{i-2^n} a_i\cos^i\theta}\prod\frac{\mathrm{d}k_i^2}{(1-y_i^2)(k_i^2-k^2)}\,.$$

As $|\cos\theta|\to\infty$ and if *n*one of the a_i's or b_i's are infinite, then the leading term in the denominator is $\cos^{2^n}\theta$ and in the numerator $\cos^{2^n-2}\theta$ this making the integral vanish. If on the other hand, one of the b_i's becomes infinite, it is easy to show that a corresponding a_{i+1} must also be infinite so that there is one extra power of $\cos\theta$ in the denominator again making the integral zero.

RIASSUNTO (*)

Si considera lo scattering da parte di una data classe di potenziali comprendenti forze di scambio. È possibile provare che, per una determinata energia, l'ampiezza dello scattering è analitica nel piano complesso $\cos\theta$ entro l'ellisse di Lehmann e che ciascun termine della serie di Born possiede i tagli e il comportamento analitico predetto da Mandelstam per il caso di un campo teorico. Si scrive una rappresentazione di Mandelstam per ciascun termine della serie di Born.

(*) *Traduzione a cura della Redazione.*

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On Scale Transformations.

J. Wess

CERN - Geneva

(ricevuto il 7 Luglio 1959)

Summary. — Invariance against scale transformation was used by W. Heisenberg [1] to define a lepton number. Some properties of this transformation are more easily investigated with scalar fields, the mass dependence of which is known. For the scalar fields and the harmonic-oscillator, some properties of the infinitesimal transformation are listed here.

The field equation

$$\partial^\mu \partial_\mu \varphi(x, m) - m^2 \varphi(x, m) = 0 \tag{1}$$

and the commutation relation

$$[\varphi(x, m), \varphi(y, m)] = i\,\Delta(x - y, m) \tag{2}$$

are invariant against the transformation

$$\left\{\begin{array}{l} x \to x' = \dfrac{1}{\lambda}\, x\,, \\ m \to m' = \lambda m\,, \\ \varphi(x, m) \to \varphi'(x', m') = \lambda\, \varphi(xm)\,, \\ \text{or} \\ \varphi'(xm) = \lambda \varphi\left(\lambda x, \dfrac{m}{\lambda}\right), \end{array}\right. \tag{3}$$

in the sense that

$$\partial^\mu \partial_\mu \varphi'(x, m) - m^2 \varphi'(xm) = 0 \tag{1a}$$

and

$$[\varphi'(xm), \varphi'(y, m)] = i\,\Delta(x-y,\, m) \tag{2a}$$

so that we might expect that there exists a unitary transformation matrix U

$$U\varphi(x, m)U^{-1} = \varphi'(x, m)\,. \tag{4}$$

If we develop $\varphi(x, m)$ in a Fourier integral:

$$\varphi(x, m) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{\mathrm{d}^3k}{\sqrt{2k_0}}\left[a(\boldsymbol{k})\exp[-ikx] + a^*(\boldsymbol{k})\exp[-ikx]\right] \tag{5}$$

with

$$k_0 = \sqrt{\boldsymbol{k}^2 + m^2}\,.$$

the $a(\boldsymbol{k})$ should only depend on $\boldsymbol{k}$, and not on m, which means, the $a(\boldsymbol{k})$ act in the same Hilbert space for all m. The mass dependence of $\varphi(x, m)$ is then simply given through k_0.

From (5), (4) and (3) follows

$$Ua(\boldsymbol{k})U^{-1} = \frac{1}{\lambda^{\frac{3}{2}}}\,a\left(\frac{\boldsymbol{k}}{\lambda}\right). \tag{4a}$$

We see that the transformation is only a permutation of the k variables in the same Hilbert space.

For the infinitesimal transformation $\lambda = 1+\varepsilon$ we have

$$\begin{cases} x'' = (1-\varepsilon)x\,, \\ m' = (1+\varepsilon)m\,, \\ \varphi'(xm) - \varphi(xm) = \varepsilon\left(\varphi + x_\nu\dfrac{\partial\varphi}{\partial x_\nu} - m\,\dfrac{\partial\varphi}{\partial m}\right) = \delta_0\varphi\,, \end{cases} \tag{3a}$$

and with

$$\begin{aligned} U &= 1 + i\varepsilon F\,, \\ [a(\boldsymbol{k}), F] &= -i\left(\tfrac{3}{2}a(\boldsymbol{k}) + \boldsymbol{k}\,\mathrm{grad}_k\, a(\boldsymbol{k})\right), \\ [a^*(\boldsymbol{k}), F] &= -i\left(\tfrac{3}{2}a^*(\boldsymbol{k}) + \boldsymbol{k}\,\mathrm{grad}_k\, a^*(\boldsymbol{k})\right). \end{aligned}$$

We wish now to compute F in terms of the field operators with the help of the action principle,

$$\delta W = F(\sigma) - F(\sigma_0)\,, \tag{6}$$

where

$$W = \int_{\Omega} \mathscr{L}(x, m)\, \mathrm{d}^4x \tag{7}$$

and

$$\delta W = \int_{\Omega'} \mathscr{L}'(x', m')\, \mathrm{d}^4x' - \int_{\Omega} \mathscr{L}(x, m)\, \mathrm{d}^4x\,. \tag{6a}$$

If we use

$$\mathscr{L}'(x'm') = \mathscr{L}'(xm) + \delta x \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}x} + \delta m \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}m} \tag{8}$$

we find

$$\delta W = \left(\int_{\sigma} - \int_{\sigma_0} \right) \mathrm{d}\sigma_\mu \left(\frac{\partial \mathscr{L}}{\partial \varphi_\mu}\, \delta_0 \varphi + \mathscr{L}\, \delta x^\mu \right) + \delta m \frac{\mathrm{d}}{\mathrm{d}m} \int_{\Omega} \mathscr{L}\, \mathrm{d}^4x\,. \tag{6b}$$

$\delta_0\varphi$ is given by (3a) and the φ satisfy the eq. (1).

The lagrangian for a scalar field is

$$\mathscr{L} = -\tfrac{1}{2}(m^2\varphi^2 - \varphi_\lambda \varphi^\lambda) \tag{9}$$

and δW becomes for a volume Ω, which is the whole space between two planes $x_0 = x_0 = \text{const}$ and $x_0 = x_0' = \text{const}$

$$\delta W = \left(\int_{x_0} - \int_{x_0'} \right) \mathrm{d}^3x \left(-\mathscr{L} x_0 + \frac{\partial \varphi}{\partial x_0} \left(\varphi + \frac{\partial \varphi}{\partial x_\nu}\, x_\nu - m \frac{\partial \varphi}{\partial m} \right) \right) + m \frac{\mathrm{d}}{\mathrm{d}m} \int_{\Omega} \mathscr{L}\, \mathrm{d}^4x\,. \tag{10}$$

In the momentum space, we can carry out the mass differentiation and we get:

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_{\Omega} \mathscr{L}\, \mathrm{d}^4x = -\frac{m x_0}{2} \int \frac{\mathrm{d}^3k}{k_0} \left[a(\boldsymbol{k})\, a(-\boldsymbol{k}) \exp\left[-2ik_0x_0\right] + \right. \tag{11}$$

$$\left. + a^*(\boldsymbol{k})\, a^*(-\boldsymbol{k}) \exp\left[2ik_0x_0\right] \right] \Big|_{x_0'}^{x_0}$$

and finally

$$F(x_0) = \frac{i}{2} \int \mathrm{d}^3k \left\{ a(\boldsymbol{k})\, a(-\boldsymbol{k}) \exp\left[-2ik_0x_0\right] \left(\frac{3}{2} - i \frac{x_0}{k_0}\, \boldsymbol{k}^2 \right) + \right. \tag{10a}$$

$$+ a^*(\boldsymbol{k})\, a^*(-\boldsymbol{k}) \exp\left[2ik_0x_0\right] \left(-\frac{3}{2} - i \frac{x_0}{k_0}\, \boldsymbol{k}^2 \right) +$$

$$+ \boldsymbol{k} \left[\exp\left[-2ik_0x_0\right] (\mathrm{grad}_k\, a(\boldsymbol{k}))\, a(-\boldsymbol{k}) - \exp\left[2ik_0x_0\right] (\mathrm{grad}_k\, a^*(\boldsymbol{k}))\, a^*(-\boldsymbol{k}) - \right.$$

$$\left. \left. - (\mathrm{grad}_k\, a(\boldsymbol{k}))\, a^*(\boldsymbol{k}) + (\mathrm{grad}_k\, a^*(\boldsymbol{k}))\, a(\boldsymbol{k}) \right] \right\}.$$

It can easily be shown that (4*b*) is satisfied. We also see that F' is not diagonal in the particle number $N_k = a(\boldsymbol{k})a^*(\boldsymbol{k})$, although $\int N_k \, \mathrm{d}^3k$ commutes with F.

Under the condition that the expression (11), integrated over the mass, gives zero, HEISENBERG derived in his paper [1], a constant of motion, which in our case is $\int (\mathrm{d}m/m) F(x_0)$. Generally, this is only constant when we integrate from $-M$ to $+M$, but then in our case also this constant of motion vanishes, since $F(x_0)$ is an even function in m.

For the harmonic oscillator with the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\dot{q}^2 - \omega^2 q^2) \,, \tag{12}$$

$$q = \frac{1}{\sqrt{2\omega}} \left[a \exp[i\omega t] + a^* \exp[-i\omega t] \right] , \tag{13}$$

where a is independent of ω, we can define the scale transformation:

$$\begin{cases} t' = \lambda t \,, \\ \omega' = \dfrac{1}{\lambda} \omega \,, \\ q'(t', \omega') = \sqrt{\lambda}\, q(t, \omega) \,, \end{cases} \tag{14}$$

which clearly leaves the commutator relation $[q, \dot{q}] = i$ invariant. Here we have

$$\delta q_0 = q'(t, \omega) - q(t, \omega) = \varepsilon \left(\frac{1}{2} q - t\dot{q} + \omega \frac{\partial q}{\partial \omega} \right) = 0 \,; \tag{15}$$

this follows from (13).

Also from the action principle, we find $F = 0$ or $U = 1$.

For the harmonic oscillator we can also define the transformation

$$\begin{cases} t' = \lambda t \,, \\ q'(t') = \sqrt{\lambda}\, q(t) \,, \quad \delta_0 q = q'(t) - q(t) = \varepsilon(\tfrac{1}{2} q - t\dot{q}) \,, \end{cases} \tag{16}$$

which also leaves the commutator invariant, and we can try to find the matrix

$$U q(t) U^{-1} = q'(t) \,, \tag{17}$$

$$i[F, q] = \tfrac{1}{2} q - t q' \qquad \text{with } U = 1 + i\varepsilon F. \tag{17a}$$

---

[1] H. P. DÜRR, W. HEISENBERG, H. MÜLLER, S. SCHLIEDER and K. YAMOZOKI: *Zur Theorie der Elementarteilchen*, in *Zeits. f. Naturfor.*

With the help of the action principle we find

$$F = -\tfrac{1}{2}(\dot{q}^2 - \dot{q}q + \omega t q^2) \,. \tag{18}$$

The transformation (17) would, for a scalar field, correspond to the transformation

$$U\varphi(x, m)U^{-1} = \varphi'(x, m') \tag{19}$$

but this matrix U does not exist, as was shown by J. M. JAUCH [2].

So we see that in the case where we have only a permutation of the e variables in the same Hilbert space, we can deduce from the action principle the infinitesimal generator F, eq. (10a). However, this F does not yield a conservation law, nor does it commute with the particle number. In the other case where the transformation would connect fields with different mass, the transformation does not exist at all.

* * *

The author would like to express his thanks to Professor C. J. BAKKER and CERN for the hospitality extended to him. He is grateful to Professor M. FIERZ, Professor W. THIRRING and Professor J. M. JAUCH for useful comments and suggestions.

(2) J. M. JAUCH: *On Scale Transformations*, not published.

RIASSUNTO (*)

Per definire un numero leptonico, W. HEISEMBERG si è servito dell'invarianza rispetto alla trasformazione scalare. Alcune proprietà di tale trasformazione sono esaminate in maniera più semplice facendo uso di campi scalari di cui sia nota la dipendenza dalla massa. Si elencano qui, per i campi scalari e per l'oscillatore armonico, alcune proprietà della trasformazione infinitesimale.

(*) *Traduzione a cura della Redazione.*

Absorptions of K^- Mesons at Rest in Light and Heavy Nuclei of the Emulsion.

C. Grote, I. Hauser, U. Kundt, U. Krecker, K. Lanius, K. Lewin and H. W. Meier

Deutsche Akademie der Wissenschaften zu Berlin
Kernphysikalisches Institut, Zeuthen - Berlin

(ricevuto il 10 Luglio 1959)

Summary. — 1000 K^- captures at rest have been analysed in an emulsion stack exposed to the Berkeley Bevatron. In $(38.4 \pm 2)\%$ of the events pions were emitted. The K^- captures we have separated into those in the light and those in the heavy nuclei of the emulsion. 25% of the events were captures in light nuclei.

Introduction.

Many works give information concerning the interaction of K^--mesons in nuclear emulsions (see *e.g.* ([1-6]) and ([15])). Until now there is no work investigating experimentally K^- captures separated in light and heavy emulsion nuclei.

([1]) European Collaboration: *Nuovo Cimento*, **12**, 91 (1959).

([2]) G. L. Bacchella, A. Berthelot, A. Bonetti, O. Goussu, F. Lévy, M. René, D. Revel, J. Sacton, L. Scarsi, G. Tagliaferri and G. Vanderhaeghe: *Nuovo Cimento*, **8**, 215 (1958).

([3]) G. B. Chadwick, S. A. Durroni, P. B. Jones, J. W. G. Wignall and D. H. Wilkinson: *Phil. Mag.*, **3**, 1193 (1958).

([4]) Y. Eisenberg, W. Koch, M. Nicolić, M. Schneeberger and H. Winzeler: *Nuovo Cimento*, **11**, 351 (1959).

([5]) European Collaboration: *International Conference on Mesons and Recently Discovered Particles* (Padova-Venezia, 1957), p. II-1.

([6]) W. Alles, N. N. Biswas, M. Ceccarelli and J. Crussard: *Nuovo Cimento*, **6**, 571 (1957).

We examined an unbiased sample of 1000 K^- absorptions at rest in nuclear emulsion, in order to determine the frequency of emitted charged reaction products, especially the frequency of charged π-mesons. The observed events have been separated into captures in light and heavy nuclei, using the different Coulomb potential barriers and the different frequencies of emitted Auger electrons in both classes of nuclei.

1. – Experimental method.

An Ilford G-5 emulsion stack consisting of 10 cm $\times$ 10 cm pellicles, being 600 μm thick, was exposed to the K^- beam of the bevatron. 45 plates of this stack we got from the Institute of Physics of the Academy of the Roumanian Peoples Republic (*).

A stripe perpendicular to the beam direction with 3 cm distance from the entrance edge of the stack was scanned for grey tracks, and all these tracks were followed to their end. Only such endings, whose distances from one of the emulsion surfaces was more than 23 μm, were taken into account. The K_ϱ-endings we have separated from protons by ionization and range measurements. Tracks with ranges $R < 5$ μm were not counted as prongs. Tracks with $R < 2.4$ μm have been classified as blobs, which are caused partly by slow Auger electrons with kinetic energies < 13 keV and partly by recoil tracks of the remaining nucleus. Tracks with $2.4 \leqslant R < 5$ μm were counted as recoil of the residual nucleus.

1000 K^- captures at rest were selected in such a way.

2. – Pion emission.

All K^- captures were carefully investigated for emission of charged pions. Pions, which did not come to rest within the stack, were identified by ionization measurements. Tracks with a grain density $g \leqslant 30$ grains/100 μm were counted as pions, since protons with these grain densities have kinetic energies $\geqslant 320$ MeV, however, the emission of a proton with such a high energy cannot be induced by one of the known K^- capture reactions. Tracks with grain densities $g > 30$ leaving the stack were identified by grain density measurements at different ranges.

349 pions were identified by the described methods. In two cases the available track length was too short for identification.

(*) We are very much indebted to the Roumanian Academy of Science. Especially Dr. E. M. FRIEDLÄNDER, who placed the plates at our disposal.

For all pion tracks the angle to the plane of the emulsion was measured. Table I shows the distribution of these angles:

TABLE I.

Dip of the pion	$0° \div 14.5°$	$14.5° \div 30°$	$30° \div 48.6°$	$48.6° \div 90°$
No. of events	98	94	80	77

If we assume that the pions are emitted isotropically, obviously we overlooked some pions with large dip. Employing a corresponding correction we get for the percentage of emitted pions a value of $(38.4 \pm 2)\%$.

Within the limits of error this value is in good agreement with the result recently published by the « European Collaboration » [1] and like this it lies considerably higher than all values measured formerly [2-5].

3. – Hyperon and hyperfragment emission.

All secondary tracks not being pion tracks were followed to the points where the particles came to rest, disintegrated, interacted or left the stack ($\sim 4\%$). Table II contains the observed hyperons:

TABLE II.

$\Sigma^+ \to p + \pi^0$ at rest	$\Sigma^+ \to p + \pi^0$ in flight	$\Sigma^+ \to \pi^+ + n$ at rest	$\Sigma^\pm \to \pi^\pm + n$ in flight	Σ^- captures at rest with $\geqslant 1$ prong
19	5	25	23	18

These values are within the statistical limits of error in agreement with the hyperon frequencies observed by other authors [2-5]. The efficiency for the observation of $\Sigma^+ \to p + \pi^0$ disintegrations is certainly 100%. Probably we have overlooked $\sim 30\%$ of the $\Sigma^+ \to \pi^+ + n$ events, because the ends of the secondary prongs were investigated only once. Furthermore we must take into account that the number of Σ^- captures without visible prongs is nearly by a factor 2 higher than the number of captures with $\geqslant 1$ prong [6]. We estimate that about $(15 \div 20)\%$ of all K^- captures are connected with Σ hyperon emission. We cannot give an exact value because of the uncertainty of the different corrections.

Furthermore we observed 40 double stars, whose connecting tracks were too short for a reliable identification. The predominant part of these tracks

was caused by hyperfragments (Hfr), but it is possible that some Σ^- hyperons with small energy are among them.

Table III contains all observed frequencies of the tracks of the 1000 analysed K^- captures at rest:

TABLE III.

No. of prongs	0	1	2	3	4	5	6	7	8	9	total
No. of captures	127	236	247	179	104	55	43	6	2	1	1000
π without Σ and double star	—	93	92	50	21	19	14	2	—	1	292
$\pi+\Sigma$	—	—	34	1	3	1	—	—	—	—	39
π+double star	—	—	3	6	3	2	4	—	—	—	18
Σ without π	—	12	10	10	12	5	2	—	—	—	51
Double star without π	—	—	5	6	5	1	4	1	—	—	22

4. – Separation of the K^- captures into light and heavy nuclei.

4·1. *Potential barrier of the heavy nuclei.* – At first we consider all σ_K stars in which at least one stable prong or Hfr was emitted. The observed capture stars originated partly from the disintegration of light nuclei of the emulsion (C, N and O) and partly from the disintegration of heavy nuclei (Ag and Br). For the separation of the two groups of stars we use the different level of the Coulomb potential barrier of light and heavy nuclei for the emission of α-particles.

If one studies the energy distribution of the α-particles arising from disintegrations with small excitation energy, one finds two maxima, which are separated from each other by a minimum. The separating method consists in classifying as disintegration of light nuclei all stars with at least one α-particle, whose energy is less than the value at the minimum.

For stars, which have at least one stable prong or a hyperfragment, we have measured the range R of all tracks with $5\ \mu m \leqslant R \leqslant 110\ \mu m$. Fig. 1 gives the range distribution of these prongs for the one and two prong σ_K stars and Fig. 2 that for all capture stars. On the basis of these distributions we have taken 8 MeV for the energy value at the minimum corresponding to a range of 41 μm. This value is in good agreement with the results found by

investigation of π^- capture stars and of disintegrations of nuclei by fast nucleons ([7-11]).

Using this method we assume that the number of protons with energies < 2.1 MeV (corresponding $R < 41$ μm) is neglegible compared with the number of α-particles. A comparison of the energy spectra and emission probabilities of both kinds of particles shows that this assumption is justified for the considered excitation energies. Comparing the energy spectrum of the α-particles corresponding to the dotted curve in Fig. 2 with the measured energy spectra of α-particles from disintegrations of light emulsion nuclei ([9]), one finds good agreement. This means that the number of protons from disintegrations of light nuclei is small also for the range interval $41 < R \leqslant 110$ μm. Since the potential barrier for the emission of protons by heavy nuclei has a value of ~ 3.5 MeV, such protons cannot exist within the investigated range interval.

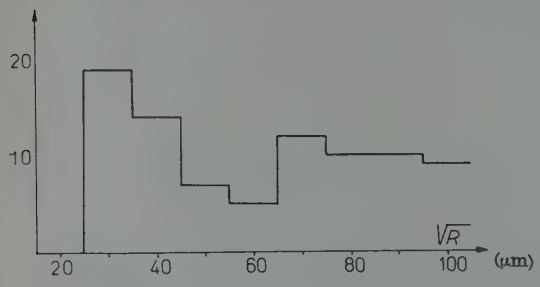

Fig. 1. – $\sqrt{R}$ distribution of the stable prongs (or Hfr) belonging to the 1- and 2-prong K^- capture stars.

We have classified all stars with $\leqslant 8$ prongs and at least one stable prong (or Hfr) in the range interval $5 \leqslant R < 41$ μm as disintegrations of light nuclei.

Previous experiments, have shown that this separation method is reliable

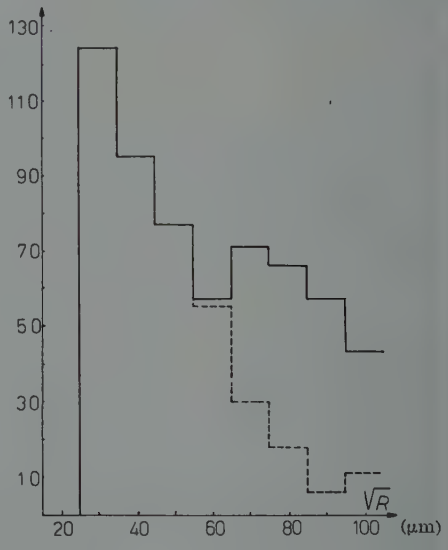

Fig. 2. – $\sqrt{R}$ distribution of the stable prongs (or Hfr) belonging to all K^- capture stars. The dotted curve gives the $\sqrt{R}$ distribution of stable prongs from such stars which have at least one prong in the range interval 5 μm $\leqslant R < 41$ μm.

([7]) M. G. K. MENON, H. MUIRHEAD and O. ROCHAT: *Phil. Mag.*, **41**, 583 (1950).
([8]) V. DE SABATA, E. MANARESI and G. PUPPI: *Nuovo Cimento*, **10**, 1704 (1953).
([9]) M. DEMEUR, A. HULEUX and G. VANDERHAEGHE: *Nuovo Cimento*, **4**, 509 (1956).
([10]) P. E. HODGSON: *Phil. Mag.*, **45**, 190 (1954).
([11]) K. LANIUS: *Nucl. Phys.*, **3**, 391 (1957).

enough, to make possible an investigation of the main characteristics of captures in heavy and light emulsion nuclei ([12]).

In this way we have identified 222 σ_K stars as captures of negative K^--mesons in light emulsion nuclei (see Table IV).

TABLE IV.

No. of prongs	1	2	3	4	5	6	7	total
No. of captures	9	38	76	53	29	17	—	222
π without Σ and double star	—	8	19	15	13	8	—	63
$\pi+\Sigma$	—	—	1	3	—	—	—	4
π+double star	—	—	4	3	2	3	—	12
Σ without π	—	1	4	8	5	—	—	18
Double star without π	—	1	3	1	—	1	—	6

4·2. *Associated Auger electrons.* – The potential barrier method fails, if one wants to classify the following 3 groups of events:

1) K_e events,

2) σ_K stars with one prong, caused by an emitted pion or $\Sigma^\pm$ hyperon,

3) σ_K stars with two prongs caused by an emitted pion and an emitted $\Sigma^\pm$ hyperon.

We have observed 266 events belonging to these three groups. Estimating the number of emitted $\Sigma^\pm$ hyperons as 17% we get as corrected number 307 events of this kind.

In order to separate these three groups of events into captures in light and heavy nuclei we have looked at all K^- captures for Auger electrons and blobs.

Theoretical ([13]) and experimental ([9,14]) investigations have shown that the mesonic Auger effect is much more frequent for the heavy emulsion atoms than for the light ones.

([12]) G. BROWN and I. S. HUGHES: *Phil. Mag.*, **2**, 777 (1957).
([13]) E. H. S. BURHOP: *The Auger Effect* (Cambridge, 1952), chap. 7.
([14]) E. B. CHESIK and J. SCHNEPS: *Phys. Rev.*, **112**, 1810 (1958).

Investigating the 1000 K^- captures we have found 282 Auger electrons with energies $\geqslant 13$ keV and 477 blobs.

A possible source of error are casual coincidences of slow background electrons with the ends of K^--mesons. We investigated 110 proton ends for « pseudo Auger electrons » and « pseudo blobs » and observed 3 electrons and 11 blobs.

In Table V the corrected frequencies of the emitted Auger electrons and blobs are given for the different classes of captures:

TABLE V.

	No. of captures	No. of Auger electrons (%)	No. of blobs (%)
Captures in light nuclei	222	3.6	7.6
Captures in heavy nuclei	471	33.0	46.6
Captures in the groups 1-3	307	29.9	45.6
All captures	1000	25.5	37.7

Obviously most of the captures belonging to the groups 1-3 occured in heavy emulsion nuclei. This result is in good agreement with the observations of GILBERT *et al.* ([15]).

5. – Discussion.

Taking into account the number of Auger electrons in the three different classes, we calculated that $\sim 10\,\%$ of the captures belonging to the groups 1-3 occured in light nuclei. Thus we get $25\,\%$ for the entire number of K^- captures in light emulsion nuclei and $75\,\%$ in the case of heavy nuclei.

For the corrected relative frequency of the emitted pions in both classes we have:

$$\text{light nuclei: } 41\,\%, \qquad \text{heavy nuclei: } 37\,\%.$$

Assuming that all captures of the groups 1-3 occur in heavy nuclei, we get

([15]) F. C. GILBERT, C. E. VIOLET and R. S. WHITE: *Phys. Rev.*, **107**, 228 (1957).

the following values:

light nuclei: 39 %, heavy nuclei: 38 %.

Investigating the observed charged Σ hyperons we get for the relative frequencies in both classes

light nuclei: 10 %, heavy nuclei: 9 %.

If we calculate the corresponding values for the double stars caused predominantly by hyperfragments, we obtain:

light nuclei: 7 %, heavy nuclei: 3 %.

A comparison of the values shows that the difference of the relative frequencies of pions and Σ hyperons emitted from light and heavy nuclei is small. On the other hand the number of emitted hyperfragments from light nuclei is two times higher than the number emitted from heavy nuclei.

* * *

We are especially grateful to Mr. W. BERNARD, Miss I. KUSSATZ, Mr. P. LANDROCK, Mrs. G. MANSKE, Mrs. R. MÜLLER, Mrs. S. STÖSSER and Mrs. A. VOIGTLÄNDER for their constant assistance throughout this work. We are indebted to Prof. M. DANYSZ and Mr. E. MARQUIT for their stimulating discussions.

RIASSUNTO (*)

In un pacchetto di emulsioni esposto al Bevatrone di Berkeley, sono state analizzate 1000 catture K^- a riposo. Nel $(38.4 \pm 2)\%$ degli eventi sono stati emessi pioni. Sono state separate le catture K^- dovute a nuclei leggeri dell'emulsione da quelle dovute a nuclei pesanti: il 25% degli eventi è costituito da catture da parte di nuclei leggeri.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIV, N. 3 1° Novembre 1959

The Field Theoretic Definition of the Nuclear Potential - I.

J. M. Charap (*) and S. P. Fubini

CERN - Geneva

(ricevuto il 14 Luglio 1959)

Summary. — It is shown that the static limit ($\mu/M \to 0$) leads to completely unreliable results in a perturbation theory definition of the potential, even in graphs without elastic scattering intermediate states. However, it is still possible to define a potential which is an energy-independent function of r alone, which when inserted into a non-relativistic Schrödinger equation reproduces the relativistic field-theoretic scattering matrix at sufficiently low energies. A method is developed whereby such a potential, in which both μ and M occur as parameters, is defined unambiguously for the case of a proton and a neutron scattering through a neutral meson field. By using dispersion relations for proton-neutron scattering, it is concluded that the potential reproduces the scattering matrix for momenta $\ll \sqrt{M\mu}$. Finally, a general method for explicit construction of the potential as a superposition of Yukawa potentials of different masses is proposed, and the one- and two-meson exchange contributions to the potential are evaluated.

Introduction.

The concept of a static potential has played a very important rôle in the theory of nucleon-nucleon scattering [1]. Almost all investigations on the subject were centered on the derivation from field theory of some kind of nuclear potential. However, there is still disagreement on the proper form of the nucleon-nucleon potential; the only universally accepted result is the one-pion exchange potential derived by Yukawa in 1935.

(*) DSIR Research Student on leave of absence from Trinity College, Cambridge.

[1] An excellent account of the subject can be found in *Suppl. Prog. Theor. Phys.*, No. 3 (1956).

A possible definition of the nuclear potential is obtained by considering the self-energy of two fixed nucleons, a distance r apart, interacting with a meson field. This definition is quite unambiguous but it is probably not legitimate to insert such a potential in a Schrödinger equation in which the two nucleons are allowed to move.

Therefore the problem arises of including $1/M$ corrections and it is just on that problem that no clear conclusion has yet been reached. We want to point out that the situation of the present theory is even worse. Indeed, a calculation of fourth order relativistic perturbation graphs shows that the μ/M expansion is divergent and therefore the static limit of such graphs is completely misleading. A similar situation is met for the electromagnetic structure of the nucleon; the physical reason for the unexpectedly large size of the recoil effects is explained in a recent paper by D. WALECKA [2]. It is to be noted that the static limit is bad for all fourth order graphs, even for those for which no ambiguity in the definition of potential existed.

In such a situation we feel that the old method of defining a potential using more or less explicit expansion in $1/M$ has to be abandoned. In this paper we consider the problem of nuclear potentials from a completely different point of view.

Consider the nucleon-nucleon scattering matrix from meson theory: such a matrix is unambiguously defined and (disregarding the existence of bound states) is the only quantity of physical interest. We shall define the potential by requiring that it shall, on insertion into a Schrödinger equation, reproduce the relativistic S-matrix in a certain energy range.

We have therefore to discuss two main questions: first, how to define unambiguously a potential depending on $r\,\mu$ and M; secondly, to prove that the potential is able to reproduce the relativistic S-matrix in a sufficiently large energy range. We think that a potential is a useful concept only if the energy range extends between zero and a momentum such that $qR \gg 1$ where R is the range of the nuclear forces (*).

In order to discuss such a question in the most simple framework, we shall discuss in the first part of this paper the case of the interaction of two scalar nucleons (proton and neutron) through a scalar meson field. This model avoids all difficulties connected with exchange forces.

In the second part, the realistic case of nucleon-nucleon interactions through a pseudoscalar meson field will be discussed.

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(2) G. F. CHEW, R. KARPLUS, S. GASIOROWICZ and F. ZACHARIASEN: *Phys. Rev.*, **110**, 265 (1958); P. FEDERBUSH, M. GOLDBERGER and S. TREIMAN: *Phys. Rev.*, **112**, 642 (1958); S. DRELL: *Proceedings of the 8th Annual Conference on High Energy Physics* (CERN, Geneva, 1958); D. WALECKA: *Nuovo Cimento*, **11**, 821 (1959).

(*) This point was made to us by Dr. Y. YAMAGUCHI.

1. – Kinematics and notation.

We shall be concerned with the scattering of two particles from a state with 4-momenta (*) $p_1 n_1$ to a state with 4-momenta $p_2 n_2$ (Fig. 1). It may be convenient to think of the particles as a proton and a neutron interacting through a neutral pion field, although for simplicity all particles be will treated as scalar.

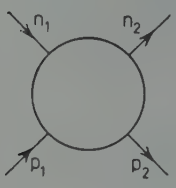

Fig. 1. – Diagram of proton-neutron scattering.

Conservation of 4-momentum gives

$$p_1 + n_1 = p_2 + n_2 \,. \tag{1.1}$$

In the case of one-meson exchange, the 4-momentum of the meson will be written 2Δ, and in general Δ will be one-half of the 4-momentum transfer. We shall make use of the following relations and definitions

$$\left\{\begin{aligned}
&\Delta = \tfrac{1}{2}(p_2 - p_1) = \tfrac{1}{2}(n_1 - n_2)\\
&P = \tfrac{1}{2}(p_1 + p_2)\\
&N = \tfrac{1}{2}(n_1 + n_2)\\
&P\cdot\Delta = 0 = N\cdot\Delta\\
&P^2 = -(M^2 + \Delta^2) = N^2\\
&C = \tfrac{1}{2}(P + N)\\
&K = \tfrac{1}{2}(P - N)\\
&C\cdot K = \Delta\cdot K = \Delta\cdot C = 0\\
&-s = C^2 = -(M^2+\eta^2) = -E^2\\
&-t = \Delta^2 = \tfrac{1}{2}\eta^2(1 - \cos\vartheta)\\
&-u = K^2 = \tfrac{1}{2}\eta^2(1 + \cos\vartheta)\\
&s + t + u = M^2.
\end{aligned}\right. \tag{1.2}$$

Here we have introduced η, the modulus of the 3-momentum of either nucleon in their centre of mass frame, and ϑ, the scattering angle in this

(*) We shall use ordinary type-face for 4-vectors, p, and bold-face for 3-vectors, $\boldsymbol{p}$. The metric is such that $A\cdot B \equiv \boldsymbol{A}\cdot\boldsymbol{B} - A_0 B_0$.

frame. Writing

$$\boldsymbol{k}_1 = \tfrac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{n}_1)$$

$$\boldsymbol{k}_2 = \tfrac{1}{2}(\boldsymbol{p}_2 - \boldsymbol{n}_2)$$

the Feynmann amplitude G is defined in terms of the field-theoretic S-matrix by

$$\langle p_2 n_2 | S | p_1 n_2 \rangle = \delta^{(3)}(\boldsymbol{p}_2 - \boldsymbol{p}_1)\, \delta^{(3)}(\boldsymbol{n}_2 - \boldsymbol{n}_1) - \tag{1.3}$$
$$- 2\pi i\, \delta^{(4)}(p_2 + n_2 - p_1 - n_1) \cdot \frac{M^2}{E^2} (\boldsymbol{k}_2 | G | \boldsymbol{k}_1) .$$

When we discuss scattering by a potential we have a scattering matrix s which is related to the T matrix by

$$(\boldsymbol{k}_2 | s | \boldsymbol{k}_1) = \delta^{(3)}(\boldsymbol{k}_2 - \boldsymbol{k}_1) - 2\pi i\, \delta(W_2 - W_1)(\boldsymbol{k}_2 | T | \boldsymbol{k}_1) \tag{1.4}$$

$$W = \frac{\boldsymbol{k}^2}{2M^*} = \frac{\boldsymbol{k}^2}{M} = \frac{\eta^2}{M}, \tag{1.5}$$

since the reduced mass $M^* = \frac{1}{2}M$.

The T matrix is then related to the potential by the integral equation

$$(\boldsymbol{k}_2 | T | \boldsymbol{k}_1) = (\boldsymbol{k}_2 | V | \boldsymbol{k}_1) - \int \mathrm{d}^3\boldsymbol{k}\, (\boldsymbol{k}_2 | V | \boldsymbol{k}) \frac{1}{W - W_1} (\boldsymbol{k} | T | \boldsymbol{k}_1) , \tag{1.6}$$

$$(\boldsymbol{k}_2 | V | \boldsymbol{k}_1) = \frac{1}{(2\pi)^3} \int \exp[-i(\boldsymbol{k}_2 - \boldsymbol{k}_1) \cdot \boldsymbol{x}] V(|\boldsymbol{x}|)\, \mathrm{d}^3\boldsymbol{x} , \tag{1.7}$$

and the original Schrödinger equation is

$$\left[-\frac{\nabla^2}{2M^*} + V(|\boldsymbol{x}|) \right] \varphi(\boldsymbol{x}) = W \varphi(\boldsymbol{x}) . \tag{1.8}$$

For a two-particle scattering there are only two linearly independent scalars, which we choose to be η^2, t writing

$$(\boldsymbol{k}_2 | G | \boldsymbol{k}_1) = G(\eta^2, t) , \tag{1.9}$$

$$(\boldsymbol{k}_2 | T | \boldsymbol{k}_1) = T(\eta^2, t) , \tag{1.10}$$

$$(\boldsymbol{k}_2 | V | \boldsymbol{k}_1) = V(t) . \tag{1.11}$$

V is independent of η^2 by virtue of the locality assumed for the potential in configuration space.

2. – Examination of the perturbation theory S-matrix.

According to our way of looking at things, the potential is to be regarded as an aid for calculating the S matrix, and is useful only in so far as it generates a T matrix which is, to some good approximation, the same as that of field theory. In order to obtain some insight into the problem, let us write down the first few terms in the perturbation theory S matrix, *viz.*, the contributions from the one- and two-meson exchange graphs illustrated in Fig. 2,

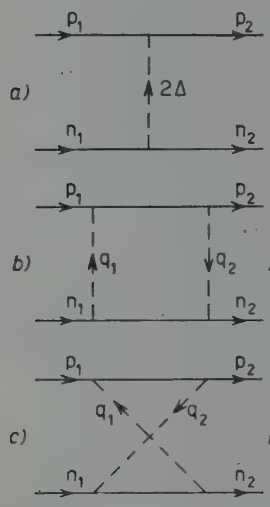

Fig. 2-*a,b,c*. – Perturbation graphs of one and two meson exchanges.

$$(2.1) \qquad G^{(1)}(\eta^2, t) = \frac{-g^2}{(2\pi)^3}\frac{1}{4M^2}\frac{1}{\mu^2 - 4t},$$

$$(2.2\mathrm{A})\ (^*) \qquad G^{(A)}(\eta^2, t) = \frac{-g^4}{(2\pi)^5}\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_0^\infty z\,\mathrm{d}z\cdot$$

$$\cdot[\mu^2(1+z) - (1-x^2)t + z^2\{M^2 - (1-y^2)s\} - i\varepsilon]^{-2},$$

$$(2.2\mathrm{B})\ (^*) \qquad G^{(B)}(\eta^2, t) = \frac{-g^4}{(2\pi)^4}\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_0^\infty z\,\mathrm{d}z\cdot$$

$$\cdot[\mu^2(1+z) - (1-x^2)t + z^2\{M^2 - (1-y^2)u\} - i\varepsilon]^{-2}.$$

$G^{(B)}$ corresponds to the crossed graph, which is universally accepted to contribute directly to the potential. Since $u = -\frac{1}{2}\eta^2(1+\cos\vartheta)$, it is neglected compared with M^2, a valid approximation, showing that $G^{(B)}$ is independent of η^2 in the adiabatic limit $(\eta/M \to 0)$. Now one has

$$G^B_{\text{adiabatic}}(\eta^2, t) = \frac{-g^4}{(2\pi)^5}\frac{1}{64M^4}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_0^\infty z\,\mathrm{d}z\cdot\left[\mu^2\left(1+\frac{z}{M}\right) - (1-x^2)t + z^2\right]^{-2}$$

and the usual next step is to go to the *static* limit $(\mu/M \to 0)$, reducing the denominator to $[\mu^2 - (1-x^2)t + z^2]^{-2}$. But this is clearly not justified since the z integration extends to ∞, and there z/M is *not* small. However, one sees that for this graph at any rate, the adiabatic limit alone is enough to

(*) See Appendix.

remove the energy dependence. The static limit is usually invoked to remove the energy dependence of the contribution arising from the uncrossed graph. We insist that it is necessary by some means to remove the energy dependence of the potential; a potential which is a function of two variables, η^2 and t, is useless since it is no simplification on the T matrix itself.

But now consider the most naïve way of defining a potential; one writes down the perturbation expansion of the T matrix arising from a potential using (6.1), and equates it term by term with that of field theory:

$$T = T^{(1)} + T^{(2)} + \dots ,$$

$$(\boldsymbol{k}_2 | T^{(1)} | \boldsymbol{k}_1) = (\boldsymbol{k}_2 | V^{(1)} | \boldsymbol{k}_1) ,$$

$$(\boldsymbol{k}_2 | T^{(2)} | \boldsymbol{k}_1) = (\boldsymbol{k}_2 | V^{(2)} | \boldsymbol{k}_1) - \int \frac{\mathrm{d}^3\boldsymbol{k} (\boldsymbol{k}_2 | V^{(1)} | \boldsymbol{k})(\boldsymbol{k} | V^{(1)} | \boldsymbol{k}_1)}{W - W_1} .$$

Proceeding with the identification, we note first that

$$V^{(1)} = T^{(1)} = G^{(1)} = \frac{-g^2}{(2\pi)^3} \frac{1}{4M^2} \frac{1}{\mu^2 - 4t} ,$$

which is already independent of η^2.

$$V^{(2)} = T^{(2)} + I ,$$

where $I =$ the iteration of the Yukawa potential $V^{(1)}$

i.e.

$$V^{(2)} = G^{(A)} + G^{(B)} + I .$$

We have already seen that $G^{(B)}$ is in the adiabatic limit independent of η^2. This is not true of $G^{(A)}$, and it is not obvious that it is true for $V^{(A)} = G^{(A)} + I$. It is at this stage that the static limit is usually invoked. However

$$G^{(A)} = \frac{-g^4}{(2\pi)^5} \frac{1}{32M^3} \frac{M}{E} \int_{-1}^{+1} \mathrm{d}x \int_0^{\infty} \mathrm{d}z \int_0^{Ez} \mathrm{d}y \, [\mu^2(1+z) - (1-x^2)t - \eta^2 z^2 + y^2]^{-2},$$

and in the adiabatic limit, we can replace E by M, to get

$$G^{(A)}_{\text{adiabatic}} = \frac{-g^4}{(2\pi)^5} \frac{1}{32M^3} \int_{-1}^{+1} \mathrm{d}x \int_0^{\infty} \mathrm{d}z \int_0^{Mz} \mathrm{d}y \, [\mu^2(1+z) - (1-x^2)t - \eta^2 z^2 + y^2]^{-2},$$

which is still dependent on η^2 as already mentioned.

$$I = \frac{g^4}{(2\pi)^6}\frac{1}{16M^3}\int \mathrm{d}^3\boldsymbol{k}\,\frac{1}{(\boldsymbol{k}_2-\boldsymbol{k})^2+\mu^2}\,\frac{1}{\boldsymbol{k}_2-\boldsymbol{k}_1^2}\,\frac{1}{(\boldsymbol{k}_1-\boldsymbol{k})^2+\mu^2} =$$

$$= \frac{g^4}{(2\pi)^5}\frac{1}{32M^3}\int_{-1}^{+1}\mathrm{d}x\int_0^\infty \mathrm{d}z\int_0^\infty \mathrm{d}y[\mu^2(1+z)-(1-x^2)t-\eta^2z^2+y^2]^{-2}\,.$$

Add these equations to obtain

$$V^{(4)} = \frac{g^4}{(2\pi)^5}\frac{1}{32M^3}\int_{-1}^{+1}\mathrm{d}x\int_0^\infty \mathrm{d}z\int_{Mz}^\infty \mathrm{d}y[\mu^2(1+z)-(1-x^2)t-\eta^2z^2+y^2]^{-2}\,.$$

And now, *still in the adiabatic limit*, it is legitimate to neglect η^2z^2 compared with $y^2 \geqslant M^2z^2$, so that

$$V^{(4)} \simeq \frac{g^4}{(2\pi)^5}\frac{1}{32M^3}\int_{-1}^{+1}\mathrm{d}x\int_0^\infty \mathrm{d}z\int_{Mz}^\infty \mathrm{d}y[\mu^2(1+z)-(1-x^2)t+y^2]^{-2}\,, \tag{2.3}$$

which is indeed independent of η^2.

We conclude that to fourth order in perturbation theory there is a cancellation which removes the energy dependence of the potential defined in this naïve way, and that this cancellation occurs in the adiabatic limit $\eta/M \to 0$, and does not require the static limit $\mu/M \to 0$, which we have seen not to be defined. In order to show that this cancellation is a very general feature of the theory, and that it occurs in all orders, it is necessary to use a more abstract approach utilising dispersion relations, and this we do in the next sections.

3. – Analytic properties of the Feynman amplitude.

Let us discuss briefly the analytic properties of the Feynman amplitude $G(\eta^2, t)$.

First of all we consider the usual kind of dispersion relation for fixed momentum transfer [3]. Since, in our simple model, a proton and an anti-

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[3] M. GOLDBERGER, Y. NAMBU and R. OEHME: *Ann. Phys.*, **2**, 226 (1957).

neutron cannot annihilate into π's, the only intermediate states contributing to the absorptive part with nucleon number zero must at least contain a nucleon-annihilation pair. So the dispersion relation has the simple form

$$(3.1)\quad G(\eta^2, t) = G(0, t) + \frac{\eta^2}{\pi}\int_0^\infty \frac{\operatorname{Im} G(\eta'^2, t)\,\mathrm{d}\eta'^2}{\eta'^2(\eta'^2 - \eta^2 - i\varepsilon)} - \frac{\eta^2}{\pi}\int_{M^2}^\infty \frac{\operatorname{Im} G^a(u't)\,\mathrm{d}u'}{(u' + t)(u' + t + \eta^2)},$$

where $G(0, t)$ is the scattering amplitude at zero kinetic energy.

The fact that no pole in η^2 appears in eq. (3.1), and that the lower limit of integration in u' is M^2 and not μ^2, is of the utmost importance for our definition of the potential.

Eq. (3.1) has a very firm theoretical basis and could be proven rigorously for reasonable values of t without the unpleasant limitation $\mu > (\sqrt{2} - 1)M$.

We now want to consider the analytic properties of $G(0, t)$ which will be directly related to the nuclear potential.

One can write

$$(3.2)\qquad G(0, t) = G(0, 0) + \frac{4t}{(2\pi)^3}\int_{\mu^2}^\infty \frac{F(\sigma^2)}{(\sigma^2 - 4t)}\frac{\mathrm{d}\sigma^2}{\sigma^2} - \frac{4t}{(2\pi)^3}\int_{M^2}^\infty \frac{F'(\sigma^2)}{\sigma^2 + 4t}\frac{\mathrm{d}\sigma^2}{\sigma^2}.$$

The representation (3.2) has not yet been proven rigorously but its validity is suggested by perturbation theory. One can easily verify it for the fourth order graphs discussed in Section 2.

The weight functions of F and F' can be related to the transition amplitudes in which the initial states are proton-antiproton and proton-antineutron respectively.

The main contributors to the r.h.s. of eq. (3.2) are the low mass states, *i.e.* the states with pions

$$(3.3)\qquad \begin{cases} F(\sigma^2) = \dfrac{-g^2}{4M^2}\delta(\sigma^2 - \mu^2) + F_2(\sigma^2) + F_3(\sigma^2) + \cdots, \\ F_n(\sigma^2) = 0 \quad \text{for} \quad \sigma^2 < (n\mu)^2. \end{cases}$$

Thus for $t \ll M^2$ we can keep only the pion contributions and write:

$$(3.4)\qquad G(0, t) \simeq G^0(t) = G(0, 0) + \frac{4t}{(2\pi)^3}\int_{\mu^2}^\infty \frac{F(\sigma^2)\,\mathrm{d}\sigma^2}{(\sigma^2 - 4t)\sigma^2}.$$

Eqs. (3.1) and (3.4) are the field theoretical basis for our definition of the potential. It is interesting to remark that the nucleon-nucleon states

affect the variable η^2 whereas the pion states give singularity only in the variable t.

Eq. (3.1) will be used in order to study the energy dependence of our potential; it will be shown that an energy independent potential can be safely used for $0 < \eta^2 \ll M\mu$.

The representation (4) will enable us to write our potential as a superposition of Yukawa potentials:

$$V(r) = \int_{\mu}^{\infty} \varrho(\sigma^2) \frac{\exp[-\sigma r]}{4\pi r} \, \mathrm{d}\sigma^2 . \tag{3.5}$$

Finally, we want to note that our starting equations (3.1) and (3.4) are both contained in the two-dimensional representation proposed by MANDELSTAM ([4]).

On the other hand, the validity of the Mandelstam representation has been recently proven in theories with potentials of the form (3.5) ([5]). Our result about the existence of a potential shows that the two simple one-dimensional relations (3.1) and (3.4) are enough to ensure the validity of the full Mandelstam representation for nucleon-nucleon scattering below 300 MeV.

4. – The definition of the potential.

We shall say that $V(|\boldsymbol{x}|)$ is a potential for the field-theoretic scattering problem considered if it leads to a T matrix such that

$$T(0, t) = G_0(t) . \tag{4.1}$$

Equation (4.1) is sufficient to lead to a determination of the potential; for KHURI ([6]) has proved for a large class of non-exchange potentials the dispersion relation at fixed momentum transfer:

$$T(\eta^2, t) = V(t) + \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} T(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon} \, \mathrm{d}\eta'^2 . \tag{4.2}$$

([4]) S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

([5]) M. GOLDBERGER *et al.* (private communication by G. F. CHEW); J. BOWCOCK and A. MARTIN: *Nuovo Cimento* (to be published).

([6]) N. N. KHURI: *Phys. Rev.*, **107**, 1148 (1957).

From this we deduce immediately

$$T(\eta^2, t) = G_0(t) + \frac{1}{\pi}\int_0^\infty \frac{\eta^2}{\eta'^2}\,\frac{\mathrm{Im}\,T(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon}\,\mathrm{d}\eta'^2\,, \tag{4.3}$$

$$V(t) = G_0(t) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,T(\eta'^2, t)}{\eta'^2}\,\mathrm{d}\eta'^2\,. \tag{4.4}$$

Furthermore the unitarity of the s matrix leads to the relation

$$i(\boldsymbol{k}_2|T^+ - T|\boldsymbol{k}_1) + 2\pi\int \mathrm{d}^3\boldsymbol{k}\,(\boldsymbol{k}_2|T^+|\boldsymbol{k})\,\delta\left(\frac{\boldsymbol{k}^2}{M} - \frac{\boldsymbol{k}_1^2}{M}\right)(\boldsymbol{k}|T|\boldsymbol{k}_1) = 0\,. \tag{4.5}$$

Equations (4.3), (4.4) and (4.5) represent our definition of the potential. One deduces T by solving the non-linear system of equations given by the dispersion relation (4.3) and unitarity (4.5). Then (4.4) leads to the potential $V(t)$ We do not yet have any unambiguous method of solving such a system, although we know that its solution is unambiguous in the framework of perturbation theory. A method for solving the system explicitly is given in the next section for the physically interesting case.

But up to this point the concept of potential has only a rather formal aspect. In order to give it a physical meaning we have to prove that T is a good approximation to G for some finite range of small η^2 (*). This can be done by comparing the dispersion relations for G and for T.

We rewrite (3.1) in the form

$$G(\eta^2, t) = G_0(t) + \frac{1}{\pi}\int_0^\infty \frac{\eta^2}{\eta'^2}\,\frac{\mathrm{Im}^{\mathrm{el}}\,G\,(\eta'^2, t)}{\eta'^2 - \eta^2 - i\varepsilon}\,\mathrm{d}\eta'^2 + 0\left(\frac{\eta^2}{\eta^2_{\mathrm{thresh}}}\right), \tag{4.6}$$

where $\mathrm{Im}^{\mathrm{el}}\,G$ is the contribution to the imaginary part of G coming from elastic processes:

$$i(\boldsymbol{k}_2|G^+ - G|\boldsymbol{k}_1)_{\mathrm{el}} + 2\pi\int \mathrm{d}^3\boldsymbol{k}\,(\boldsymbol{k}_2|G^+|\boldsymbol{k})\,\frac{M}{E}\,\delta\left(\frac{\boldsymbol{k}^2}{M} - \frac{\boldsymbol{k}_1^2}{M}\right)(\boldsymbol{k}|G|\boldsymbol{k}_1) = 0\,. \tag{4.7}$$

The remaining terms coming from inelastic processes and nucleon-antinucleon scattering are negligible for η^2 small as compared with the threshold

(*) Actually, what we want is that T should be a good approximation to the field-theoretic T matrix, but this is already well-approximated by G which differs only by a factor M^2/E^2 which is almost unity for $\eta^2 \ll M^2$.

η^2_{thresh} (we assume that the scattering angle is physical or nearly physical so that $|t| \ll \eta^2_{\text{thresh}}$ if $\eta^2 \ll \eta_{\text{thresh}}$).

The system of equations (4.6) and (4.7) is the real physical justification for the introduction of a potential in quantum field theory. They show how, for a reasonable energy range, the scattering amplitude, which is essentially $G(\eta^2, t)$, a function of two variables, is related to $G_0(t)$, a function of only one variable. The introduction of an r-dependent potential is a very convenient way of visualising this important physical property of nucleon-nucleon scattering.

To continue with our discussion of the range of validity of the potential defined above, we note that to order $\eta^2/\eta^2_{\text{thresh}}$, eqs. (4.6) and (4.7) are formally identical with (4.3) and (4.5). Thus a comparison leads to

$$G(\eta^2, t) = T(\eta^2, t) + 0\left(\frac{\eta^2}{\eta^2_{\text{thresh}}}\right) \qquad \eta^2_{\text{thresh}} = M\mu + \tfrac{1}{4}\mu^2 , \tag{4.8}$$

so that eq. (4.8) shows that the potential can safely be used in order to compute nucleon-nucleon scattering for a wide range of energies. The range of validity of the potential is, as expected, connected with the threshold energy

$$\frac{\eta^2_{\text{thresh}}}{\mu^2} \simeq 7 ,$$

which shows that the potential concept is a useful one. For the range of the forces is $\sim 1/\mu$, and if this ratio were near to unity, the most useful statement that could be made would be that of effective-range theory.

The relationship between the range of validity of an energy-independent potential and the mass spectrum is a very general one, and could in principle be applied to a wide variety of systems. A possible example is the optical model (*).

Our method has some formal analogy with that suggested by GOLDBERGER, NAMBU and OEHME [3], and discussed further by MATSUYAMA and MIYAZAWA [7]. We wish to emphasize two important differences. Firstly, the treatment in GNO is in the realistic case in which charged mesons also enter, and therefore the upper limit for nucleon-antinucleon effects is $\sim \mu^2/M$. This limits the range of validity of the potential to $\eta \ll \mu^2/M$ and as remarked above, means that the potential introduces little more than an effective-range treatment. When we shall discuss the realistic problem in Part II of this work, we shall use

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(*) We are indebted to Prof. V. F. WEISSKOPF for suggesting this remark in the course of a most illuminating discussion.

(7) S. MATSUYAMA and H. MIYAZAWA: *Prog. Theor. Phys.*, **19**, 517 (1958).

different dispersion relations which will enable us to maintain the present wide range of validity for the potential, by treating exchange and direct forces on the same ground.

A second difference is in writing Im T rather than Im G in the dispersion integral defining V (eq. (4.4)). Since this equation is an unsubtracted one, this is no insignificant difference (*).

5. – Explicit construction of the potential.

In this Section we proceed to the explicit solution of eqs. (4.3) and (4.5) in the physically interesting case for which $G_0(t)$ is of the form

$$G_0(t) = \frac{1}{(2\pi)^3}\int \frac{F(\sigma^2)}{\sigma^2 - 4t}\,\mathrm{d}\sigma^2\,, \tag{5.1}$$

$$F(\sigma^2) = \frac{-g^2}{4M^2}\,\delta(\sigma^2 - \mu^2) + F_2(\sigma^2) + F_3(\sigma^2) + \dots\,. \tag{5.2}$$

Here g is the renormalized coupling constant, and $F_n(\sigma^2)$ is obtained from the effect of n-meson exchanges between two nucleons.

We also know that

$$F_n(\sigma^2) = 0 \qquad \text{for } \sigma^2 < (n\mu)^2. \tag{5.3}$$

We make the ansatz

$$T(\eta^2 t) = \frac{1}{(2\pi)^3}\int \frac{f(\sigma^2, \eta^2)}{\sigma^2 - 4t}\,\mathrm{d}\sigma^2\,. \tag{5.4}$$

(*) A striking example of such a difference can be seen in fourth order perturbation theory.

Im G is given in eq. (6.4). Im T is simply given by (E/M) Im G. This difference would at first sight seem to be unimportant. However, we have:

$$\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, G(\eta^2 t)}{\eta^2}\,\mathrm{d}\eta^2 = G_A(0, t)\,,$$

$$\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, T(\eta^2 t)}{\eta^2}\,\mathrm{d}\eta^2 = I(0, t)\,,$$

where $I(\eta^2 t)$ is the iterated Yukawa potential. The difference $G_A - I$ is by no means small and gives an important contribution to the fourth order potential.

From eq. (4.3) we get

$$f(\sigma^2, \eta^2) = F(\sigma^2) + \frac{1}{\pi}\int_0^\infty \frac{\eta^2}{\eta'^2}\frac{\operatorname{Im} f(\sigma^2, \eta'^2)}{\eta'^2 - \eta^2 - i\varepsilon}\,\mathrm{d}\eta'^2 . \tag{5.5}$$

We also write

$$V(t) = \frac{1}{(2\pi)^3}\int \frac{\varrho(\sigma)^2}{\sigma^2 - 4t}\,\mathrm{d}\sigma^2 , \tag{5.6}$$

this being the momentum space-form of

$$V(\boldsymbol{x}) = \int \varrho(\sigma^2)\frac{\exp[-\sigma r]}{4\pi r}\,\mathrm{d}\sigma^2 , \tag{5.7}$$

showing that the potential is a superposition of Yukawa potentials corresponding to a mass σ, with the spectral function ϱ giving the spectrum of masses. It is important to observe that the Khuri result is valid for any potential with such a representation.

Corresponding to (4.4) we have

$$\varrho(\sigma^2) = F(\sigma^2) - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} f(\sigma^2, \eta^2)}{\eta^2}\,\mathrm{d}\eta^2 . \tag{5.8}$$

It is also possible to write the unitarity relation (4.5) in terms of the spectral function $f(\sigma^2, \eta^2)$

$$\operatorname{Im} f(\sigma^2, \eta^2) = \int \mathrm{d}\sigma_1^2 \int \mathrm{d}\sigma_2^2 f^*(\sigma_1^2, \eta^2)\, f(\sigma_2^2, \eta^2)\, K(\sigma_1^2, \sigma_2^2; \sigma^2; \eta^2) , \tag{5.9}$$

where the kernel

$$K = \frac{-M}{8\pi\eta}\left\{\sigma^4 - 2\sigma^2\left[\frac{\sigma_1^2\sigma_2^2}{2\eta^2} + \sigma_1^2 + \sigma_2^2\right] + (\sigma_1^2 - \sigma_2^2)^2\right\}^{-\frac{1}{2}} ,$$

$$\sigma^2 > \sigma_1^2 + \sigma_2^2 + \frac{\sigma_1^2\sigma_2^2}{2\eta^2} + 2\sigma_1\sigma_2\left\{1 + \frac{\sigma_1^2 + \sigma_2^2}{4\eta^2} + \frac{\sigma_1^2\sigma_2^2}{16\eta^4}\right\}^{\frac{1}{2}} , \tag{5.11}$$

and $K = 0$ otherwise. In particular

$$K = 0 \qquad \text{for} \quad \sigma^2 < (\sigma_1 + \sigma_2)^2 . \tag{5.12}$$

We are now in a position to give the prescription for a unique determination of the potential, or equivalently of its spectral function $\varrho(\sigma^2)$ in terms of the

spectral function $F(\sigma^2)$ which is to be obtained from field theory. From (5.2) we see that

$$F(\sigma^2) = \frac{-g^2}{4M^2}\,\delta(\sigma^2-\mu^2) \quad \text{for} \quad \sigma^2 < 4\mu^2\,,$$

$$\operatorname{Im} F(\sigma^2) = 0 \quad \text{for} \quad \sigma^2 < 4\mu^2\,,$$

so that

$$\varrho(\sigma^2) = \frac{-g^2}{4M^2}\,\delta(\sigma^2-\mu^2) \quad \text{for} \quad \sigma^2 < 4\mu^2\,, \tag{5.13}$$

from (5.8), and

$$f(\sigma^2,\eta^2) = \frac{-g^2}{4M^2}\,\delta(\sigma^2-\mu^2) \quad \text{for} \quad \sigma^2 < 4\mu^2\,, \tag{5.14}$$

from (5.5). We can now use unitarity to compute $\operatorname{Im} f(\sigma^2,\eta^2)$ for $4\mu^2<\sigma^2<9\mu^2$, since equation (5.12) shows that (5.9) then gives just

$$\operatorname{Im} f(\sigma^2,\eta^2) = \frac{g^4}{16M^4}\,K(\mu^2,\mu^2;\sigma^2;\eta^2)_{4\mu^2<\sigma^2<9\mu^2}\,, \tag{5.15}$$

and substitute this into (5.8) and (5.5), together with the values of $F(\sigma^2)$ in the interval $4\mu^2<\sigma^2<9\mu^2$ to obtain $\varrho(\sigma^2)$ and $f(\sigma^2,\eta^2)$ respectively in this same interval. Then once again the unitarity relation enables an extension of the knowledge of $\operatorname{Im} f(\sigma^2,\eta^2)$ to the interval $9\mu^2<\sigma^2<16\mu^2$, and then (5.8) and (5.5) give an extension of ϱ and of f, and so on, step by step, extending the potential to include the effects of more and more meson exchanges, *i.e.* in configuration space, to shorter and shorter ranges.

6. – Application to the one- and two-pion exchange potentials.

We shall now apply the methods of the previous section in order to derive the spectral function of the potential for $\sigma^2<9\mu^2$. According to (5.2) and (5.3) the only contributions arise from the exchange of one or two pions. These can be separated into their respective Born approximations, illustrated by the graphs of Fig. 2, which are to be calculated with the renormalized coupling constant, and what are normally called the rescattering corrections, examples of which are given in Fig. 3. We have as yet no method of calculating these corrections, although in principle one might derive them by an analytic continuation of the scattering amplitude for meson-nucleon scattering

to unphysical angles. This approach has been used for the similar problem in the nucleon electromagnetic form-factor (2). However, we feel that lacking an adequate theory of meson-meson interactions, it is premature to apply this method to the present problem.

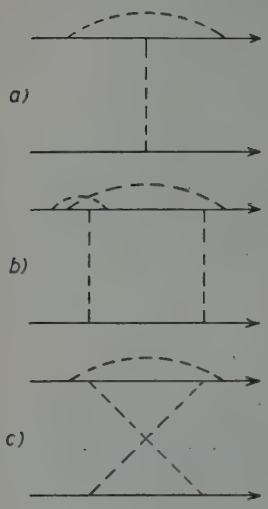

Fig. 3. – Examples of rescattering correction graphs.

In this circumstance, we shall content ourselves with writing:

$$F_2(\sigma^2) = F^{(A)}(\sigma^2) + F^{(B)}(\sigma^2) + F^{12}_{\text{rescattering}}(\sigma^2) .$$

The rescattering corrections to the graph of Fig. 2*a* (*e.g.* Fig. 3*a*) do not contribute in the interval $\sigma^2 < 9\mu^2$.

We recall the amplitudes for the perturbation graphs of Fig. 2 given in equations (2.1) and (2.2):

$$(6.1) \qquad G^{(1)}(\eta^2, t) = \frac{-g^2}{(2\pi)^3} \frac{1}{4M^2} \frac{1}{\mu^2 - 4t} ,$$

$$(6.2A)^* \quad G^{(A)}(\eta^2, t) = \frac{-g^4}{(2\pi)^5} \frac{1}{64M^2} \int_{-1}^{+1} \mathrm{d}x \int_{-1}^{+1} \mathrm{d}y \int_{0}^{\infty} z \, \mathrm{d}z \cdot$$

$$\cdot [\mu^2(1+z) - (1-x^2)t + z^2\{M^2 - (1-y^2)s\} - i\varepsilon]^{-2} ,$$

$$(6.2B)^* \quad G^{(B)}(\eta^2, t) = \frac{-g^4}{(2\pi)^5} \frac{1}{64M^2} \int_{-1}^{+1} \mathrm{d}x \int_{-1}^{+1} \mathrm{d}y \int_{0}^{\infty} z \mathrm{d}z \cdot$$

$$\cdot [\mu^2(1+z) - (1-x^2)t + z^2\{M^2 - (1-y^2)u\} - i\varepsilon]^{-2} .$$

That A and B are related by crossing symmetry is evident from the form of these expressions.

Equation (6.1) gives just the first term in (5.2). We can write down the dispersion relations for $G^{(A)}$ and $G^{(B)}$ on inspection.

$$(6.3A) \qquad G^{(A)}(\eta^2, t) = \frac{1}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{d}s'}{s' - s - i\varepsilon} F(s' - M^2, t) ,$$

$$(6.3B) \qquad G^{(B)}(\eta^2, t) = \frac{1}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{d}u'}{u' - u - i\varepsilon} F(u' - M^2, t) ,$$

(*) For the derivation of equations with an asterisk, see the Appendix.

where

$$(6.4)^* \quad F(\eta^2, t) = \operatorname{Im} G^{(A)}(\eta^2, t) = \\ = \frac{-g^4}{(2\pi)^4} \frac{1}{64 M^3} \frac{M}{E} \int_{-1}^{+1} \mathrm{d}x \, \eta [\mu^4 + 4\eta^2 \mu^2 - 4\eta^2 (1 - x^2) t]^{-1} .$$

Taking $\eta^2 = 0$ and for $t \ll M^2$ we simply have:

$$(6.5A) \quad G_0^{(A)}(t) = \frac{1}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{d}s'}{s' - M^2} F(s' - M^2, t) ,$$

$$(6.5B) \quad G_0^{(B)}(t) = \frac{1}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{d}u'}{u'} F(u' - M^2, t) ,$$

$$G_0^{(A)}(t) = \frac{-g^4}{(2\pi)^5} \frac{1}{64 M^2} \int_{-1}^{+1} \mathrm{d}x \int_{-1}^{+1} \mathrm{d}y \int_0^{\infty} z \, \mathrm{d}z [\mu^2 (1 + z) - (1 - x^2) t + z^2 M^2 y^2]^{-2} ,$$

$$G_0^{(B)}(t) = \frac{-g^4}{(2\pi)^5} \frac{1}{64 M^2} \int_{-1}^{+1} \mathrm{d}x \int_{-1}^{+1} \mathrm{d}y \int_0^{\infty} z \, \mathrm{d}z [\mu^2 (1 + z) - (1 - x^2) t + z^2 M^2]^{-2} ,$$

or after a little manipulation

$$(6.6A) \quad G_0^{(A)}(t) = \frac{-g^4}{(2\pi)^5} \frac{1}{32 M^3 \mu^2} \int_{-1}^{+1} \mathrm{d}x \int_0^{\infty} \mathrm{d}\xi \left[\xi^2 + \mu^2 + \frac{\mu^2 \xi}{M} - (1 - x^2) t \right]^{-1} ,$$

$$(6.6B) \quad G_0^{(B)}(t) = \frac{g^4}{(2\pi)^5} \frac{1}{32 M^3} \frac{\partial}{\partial \tau} \int_{-1}^{+1} \mathrm{d}x \int_0^{\infty} \mathrm{d}\xi \left[\xi^2 + \mu^2 + \frac{\tau}{M} \xi - (1 - x^2) t \right]^{-1} \Big|_{\tau = \mu^2} .$$

This means that we need to put into spectral form integrals of the kind

$$\frac{1}{(2\pi)^3} \int_0^{\infty} \mathrm{d}\xi \int_{-1}^{+1} \mathrm{d}x [\xi^2 + \mu^2 + 2\alpha\xi - t(1 - x^2)]^{-1} .$$

By a change of integration variable this is easily reduced to

$$\frac{1}{(2\pi)^3} \int_0^{\infty} \mathrm{d}\xi \int_{4(\xi^2 + \mu^2 + 2\alpha\xi)}^{\infty} \mathrm{d}\sigma^2 \frac{1}{\sigma^2 - 4t} \frac{4}{\sigma^2} \left\{ 1 - \frac{4}{\sigma^2} (\xi^2 + \mu^2 + 2\alpha\xi) \right\}^{-\frac{1}{2}} = \\ = \frac{1}{(2\pi)^3} \int_{4\mu^2}^{\infty} \mathrm{d}\sigma^2 \frac{1}{\sigma^2 - 4t} A(\sigma^2) ,$$

where

$$(6.7)\quad A(\sigma^2) = \int_0^{-\alpha+\sqrt{\alpha^2+(\sigma^2/4)-\mu^2}} \mathrm{d}\xi \frac{4}{\sigma^2}\left\{1-\frac{4}{\sigma^2}(\xi^2+\mu^2+2\alpha\xi)\right\}^{-\frac{1}{2}} = \frac{2}{\sigma}\,\mathrm{tg}^{-1}\left\{\frac{1}{2\alpha}\sqrt{\sigma^2-4\mu^2}\right\}.$$

Thus

$$(6.8A)\quad G_0^{(A)}(t) = \frac{1}{(2\pi)^3}\int_{4\mu^2}^{\infty}\mathrm{d}\sigma^2\,\frac{1}{\sigma^2-4t}\,F^{(A)}(\sigma^2),$$

$$(6.8B)\quad G_0^{(B)}(t) = \frac{1}{(2\pi)^3}\int_{4\mu^2}^{\infty}\mathrm{d}\sigma^2\,\frac{1}{\sigma^2-4t}\,F^{(B)}(\sigma^2),$$

where

$$(6.9A)\quad F^{(A)}(\sigma^2) = \frac{g^4}{64\pi^2}\,\frac{1}{M^3\mu^2}\,\frac{1}{\sigma}\left\{\Theta(\sigma)-\frac{\pi}{2}\right\},$$

$$(6.9B)\quad F^{(B)}(\sigma^2) = \frac{-g^4}{64\pi^2}\,\frac{1}{M^3\mu^2}\,\frac{1}{\sigma}\,\frac{\sin 2\Theta}{2},$$

$$(6.10)\quad \Theta(\sigma) = \mathrm{tg}^{-1}\left\{\frac{\mu}{M}\,\frac{\mu}{\sqrt{\sigma^2-4\mu^2}}\right\}.$$

Thus in the interval $4\mu^2<\sigma^2<9\mu^2$

$$(6.11)\quad F(\sigma^2) = \frac{g^3}{64\pi^2}\,\frac{1}{M^3\mu^2}\,\frac{1}{\sigma}\left\{\Theta(\sigma)-\frac{1}{2}\sin 2\Theta(\sigma)-\frac{\pi}{2}\right\} + F^{(2)}_{\text{rescattering}}(\sigma^2).$$

As stated in (5.15), for $4\mu^2<\sigma^2<9\mu^2$

$$(6.12)\quad \mathrm{Im}\,f(\sigma^2,\eta^2) = \frac{g^4}{16M^4}\,K(\mu^2,\mu^2;\sigma^2;\eta^2) = \frac{-g^4}{128M^3\pi}\,\frac{1}{\eta}\left\{\sigma^4-4\sigma^2\left(\mu^2+\frac{\mu^4}{4\eta^2}\right)\right\}^{-\frac{1}{2}} -$$

$$-\frac{1}{\pi}\int_{\mathrm{II}}^{\infty}\frac{\mathrm{Im}\,f(\sigma^2,\eta^2)\,\mathrm{d}\eta^2}{\eta^2} = \frac{g^4}{64\pi^2}\,\frac{1}{M^3}\int_0^{\infty}\frac{\mathrm{d}\eta}{\eta^2}\left\{\sigma^4-4\sigma^2\left(\mu^2+\frac{\mu^4}{4\eta^2}\right)\right\}^{-\frac{1}{2}} =$$

$$= \frac{g^4}{64\pi^2}\,\frac{1}{M^3\mu^2}\,\frac{\pi}{2\sigma}.$$

Thus in the interval $4\mu^2<\sigma^2<9\mu^2$, eq. (4.8) gives

$$(6.13)\quad \varrho(\sigma^2) = \frac{g^4}{64\pi^2}\,\frac{1}{M^3\mu^2}\,\frac{1}{\sigma}\left\{\Theta-\frac{1}{2}\sin 2\Theta\right\} + F^{(2)}_{\text{rescattering}}(\sigma^2).$$

It is interesting to note that if Θ were small, in the absence of the rescattering corrections, there would be a complete cancellation. This is nothing other than a consequence of the well known result that the Yukawa potential is a complete solution of the interaction potential for two *static* nucleons in the scalar neutral theory.

It is also clear that this method of defining the potential is related to the perturbation theory approach of Section 2, for the term (6.12) added to F to get ϱ is just the same as the zero-energy iterated Yukawa potential.

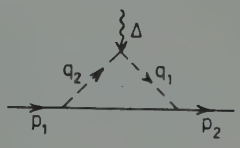

Fig. 4. – Graph contributing to form factor.

We have already remarked in Section 2 that the static limit is not defined. This is even more clear from (6.13) and (6.11). For when $\mu/M \ll 1$, in the region where $\sigma^2 \geqslant 4\mu^2$ it is still not permissible to expand (6.11) in powers of μ/M and then to ignore all but the first few terms. This feature is closely related to the similar lack of definition of the static limit for the nucleon electromagnetic form-factor, and to the similarity between the graphs of Figs. 2*b* and 2*c* with that of Fig. 4 for the form-factor. In both cases a meson is absorbed by a recoiling nucleon from a state in which all angular momentum partial waves relative to this nucleon are present. Furthermore, this interdiction against the static limit is strongest at the low mass end of the spectrum, corresponding to long-wave potentials, just where one might have hoped the static limit to be most reasonable.

Appendix

Some results used in Sections 2 and 6.

Defining $Q = \frac{1}{2}(q_1 + q_2)$ the amplitude $G^{(A)}$ arising from the graph A of Fig. 2*b* is

$$G^{(A)}(\eta^2, t) = \frac{ig^4}{(2n)^7} \frac{1}{4M^2} \int \mathrm{d}^4 Q \cdot$$

$$\cdot \frac{1}{[(Q+\Delta)^2+\mu^2][(Q-\Delta)^2+\mu^2][(P+Q)^2+M^2][(N-Q)^2+M^2)]} =$$

$$= \frac{ig^4}{(2\pi)^7} \frac{1}{4M^2} \int \mathrm{d}^4 Q \frac{3}{2} \int_{-1}^{+1} \mathrm{d}x \int_{-1}^{+1} \mathrm{d}y \int_0^1 w(1-w)\,\mathrm{d}w [Q^2 - 2S\cdot Q + \Lambda]^{-4},$$

where $S = \Delta x w - (K + Cy)(1 - w)$; $\Delta = \mu^2 w - \Delta^2(2w - 1) - i\varepsilon$

(2.2A) and (6.2A)

$$
\begin{aligned}
&G^{(A)}(\eta^2, t) = \frac{ig^4}{(2\pi)^7}\,\frac{1}{4M^2}\,\frac{3}{2}\,\frac{i\pi^2}{6}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_{0}^{1} w(1-w)\,\mathrm{d}w\,(\Delta - S^2)^{-2} = \\
&= \frac{-g^4}{(2\pi)^5}\,\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_{0}^{1} w(1-w)\,\mathrm{d}w\cdot \\
&\cdot[\mu^2 w - (1-x^2)w^2 t + (1-w)^2\{M^2 - (1-y^2)s\} - i\varepsilon]^{-2} = \\
&= \frac{-g^4}{(2\pi)^5}\,\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_{0}^{1} z\,\mathrm{d}z\cdot \\
&\cdot[\mu^2(1+z) - (1-x^2)t + z^2\{M^2 - (1-y^2)s\} - i\varepsilon]^{-2}\,.
\end{aligned}
$$

Similarly for graph B of Fig. 2c

$$
\begin{aligned}
G^{(B)}(\eta^2, t) &= \frac{ig^4}{(2\pi)^7}\,\frac{1}{4M^2}\int \mathrm{d}^4Q\cdot \\
&\cdot\frac{1}{[(Q+\Delta)^2+\mu^2][(Q-\Delta)^2+\mu^2][(P-Q)^2+M^2][(N-Q)^2+M]^2} = \\
&= \frac{ig^4}{(2\pi)^7}\,\frac{1}{4M^2}\int \mathrm{d}^4Q\,\frac{3}{2}\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}y\int_{0}^{1} w(1-w)\,\mathrm{d}w[Q^2 - 2R\cdot Q + \Delta]^{-4}\,,
\end{aligned}
$$

$$
R = \Delta x\, w + (Ky + C)(1 - w)\,,
$$

which is the same as S with K and $-C$ interchanged. Thus $G^{(B}$ is the same as $G^{(A)}$ with s and u unterchanged, leading immediately to (2.2B) and (6.2B)

(6.4)

$$
\begin{aligned}
&G^{(A)}(\eta^2, t) = \frac{-g^4}{(2\pi)^5}\,\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\,\frac{\partial}{\partial\beta}\int_{-1}^{+1}\mathrm{d}y\int_{0}^{\infty}\mathrm{d}z[\alpha z^2 - \beta z - \gamma + i\varepsilon]^{-1}\Big|_{\substack{\alpha=\eta^2-E^2y^2\\ \beta=\mu^2\\ \gamma=\mu^2-t(1-x^2)}}\,, \\
&\operatorname{Im} G^{(A)}(\eta^2, t) = \frac{-g^4}{(2\pi)^5}\,\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\,\frac{\partial}{\partial\beta}\int_{-1}^{+1}\mathrm{d}y\int_{0}^{\infty}\mathrm{d}z[(-\pi)\,\delta(\alpha z^2 - \beta z - \gamma) = \\
&= \frac{-g^4}{(2\pi)^5}\,\frac{1}{64M^2}\int_{-1}^{+1}\mathrm{d}x\,\frac{\partial}{\partial\beta}\,2\int_{0}^{\eta/E}\mathrm{d}y\,(-\pi)(\beta^2 + 4\alpha\gamma)^{-\frac{1}{2}} = \\
&= \frac{-g^4}{(2\pi)^5}\,\frac{\pi\eta}{32M^2E}\int_{-1}^{+1}\mathrm{d}x[\mu^4 - 4\eta^2(1-x^2)t + 4\eta^2\mu^2]^{-1} = \\
&= \frac{-g^4}{(2\pi)^4}\,\frac{1}{64M^3}\,\frac{M}{E}\int_{-1}^{+1}\mathrm{d}x\,\eta[\mu^4 - 4\eta^2(1-x^2)t + 4\eta^2\mu^2]^{-1}\,.
\end{aligned}
$$

RIASSUNTO

Si discute prima di tutto la validità dell'approssimazione statica ($\mu/M \to 0$) in teoria delle perturbazioni per l'urto nucleone-nucleone. Si dimostra che le correzioni in μ/M sono tanto grandi da rendere l'approssimazione non valida persino per diagrammi senza stati intermedi di scattering elastico. Tuttavia è possibile definire un potenziale locale indipendente dall'energia che, inserito in una equazione di Schrödinger non relativistica riproduce la matrice di scattering della teoria relativistica dei campi per energie sufficientemente basse. Sviluppiamo un metodo per cui tale potenziale (in cui μ ed M intervengono come parametri) è definito univocamente per scattering protone-neutrone attraverso un campo mesonico neutro. Usando le relazioni di dispersione per scattering protone-neutrone si dimostra che il potenziale riproduce la matrice di scattering per momenti $\ll\sqrt{\mu M}$. Infine si propone un metodo generale per costruire il potenziale come sovrapposizione di potenziali di Yukawa di masse diverse. Il metodo è applicato per costruire il potenziale dovuto allo scambio di uno o due mesoni.

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On the Coupling of the Elementary Particles with the Electromagnetic Field (*).

C. G. Bollini (**)

Department of Mathematics, Imperial College - London

(ricevuto il 16 Luglio 1959)

Summary. — We are trying to build up a theory in which all the supplementary conditions are considered as constraint equations imposed on the wave function and implying that only $2s+1$ (s=spin) of its components are really independent. In the already treated free field case, all the elementary particles obey the Klein-Gordon equation of motion. To introduce the electromagnetic field in a general way we consider a basic interaction involving the transformation matrix of the wave function. A process is described, by means of which a gauge invariant equation of motion can be constructed. The theory is renormalizable and describes particles having the normal gyromagnetic factor 2 for any spin value.

1. – Introduction.

In two previous papers ([1,2]) we have presented a theory for free elementary particles of any spin, in which all the so called supplementary conditions are considered as constraint equations imposed on the wave function. We here give a brief summary.

An elementary particle of spin s is represented by means of a field Ψ^a ([3]) a simbolizing either a set of s tensor indices (s = integer) or a set of $s-\frac{1}{2}$ tensor indices plus one spinor index (Rarita-Schwinger representation ([4])). The

(*) Supported by « Consejo Nacional de Investigaciones », Argentina.

(**) On leave of absence from « Comisión Nacional de Energía Atómica », Argentina.

([1]) C. G. Bollini: *Nuovo Cimento*, **11**, 342 (1959).

([2]) C. G. Bollini: *Nuovo Cimento*, **13**, 46 (1959).

([3]) We will often suppress the index a.

([4]) W. Rarita and J. Schwinger: *Phys. Rev.*, **60**, 61 (1941).

particle obeys the Klein-Gordon equation of motion

$$(-p_\mu p_\mu - m^2)\Psi^a \equiv (p^2 - m^2)\Psi^a = 0 \tag{1.1}$$

and satisfies the constraint equation

$$\Psi^a = P^{a,b}\Psi^b , \tag{1.2}$$

$P^{a,b}$ being a projection operator assuring the irreducibility of Ψ^a under the rotation group in the rest system of the represented particle ([2]).

$$P^{a,b}P^{b,c} = P^{a,c} ,$$

$$\mathrm{Tr}\,\{P\} = P^{a,a} = 2s+1 .$$

These properties and (1.2) show that Ψ^a has only $2s+1$ independent components. We now quote the form of $P^{a,b}$ for $s=\frac{1}{2}$ and $s=1$ ([2,5]).

For $s=\frac{1}{2}$,

$$P^{a,b} = \frac{1}{2}\left(\delta^{ab} - i\frac{\gamma_\mu^{ab}p_\mu}{p}\right),$$

or, suppressing the indices

$$P = \frac{1}{2}\left(1 - i\frac{\gamma\cdot p}{p}\right). \tag{1.3}$$

For $s=1$,

$$P^{a,b} = \delta^{a,b} + \frac{p^a p^b}{p^2} . \tag{1.4}$$

The aim of the present paper is to describe in a general way the interaction of the elementary particles with the electromagnetic field. To this end we will introduce in the next section the matrix describing the infinitesimal transformation properties of Ψ. This matrix will play a fundamental role in the interaction.

In Section 3, the unconstrained equation of motion is derived by the usual gauge invariant substitution performed in a particular form of the Klein-Gordon equation. An equation compatible with (1.2) is then constructed using the perturbation expansion as a tool and gauge invariance as a condition. In Section 6 we treat the $s=\frac{1}{2}$ case showing the equivalence with Dirac's theory.

([5]) For the general case see R. E. Behrends and C. Fronsdal: *Phys. Rev.*, **106**, 345 (1957). For half-integer spin, their Θ operators must be multiplied by (1.3) to obtain our P operators.

The theory is renormalizable and the normal gyromagnetic factor of the described particles is 2 for any spin value. The last property constitutes the most important physical difference with the Fierz-Pauli theory ([6]), for which this factor is the inverse of the spin ([7]).

Although the theory is not in its final form and some points remain to be clarified, it is hoped that even at this stage it may be useful enough to justify the present paper.

2. – The transformation matrix.

Under an infinitesimal Lorentz transformation

$$x'_\mu = a_{\mu\nu} x_\nu , \qquad a_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu} = \delta_{\mu\nu} - \omega_{\nu\mu}$$

the wave function transforms linearly

$$\Psi^{a'} = \tfrac{1}{4} T^{a,b}_{\mu\nu} a_{\mu\nu} \Psi^b . \tag{2.1}$$

As usual ([8]), the symmetric part of $T_{\mu\nu}$ is taken to be the metric tensor and the antisymmetric part is twice the spin matrix,

$$T^{a,b}_{\mu\nu} = \delta_{\mu\nu}\delta^{ab} + 2iS^{a,b}_{\mu\nu} . \tag{2.2}$$

For $s = \frac{1}{2}$,

$$T_{\mu\nu} = \gamma_\mu \gamma_\nu \tag{2.3}$$

and for $s = 1$,

$$T^{a,b}_{\mu\nu} = \delta_{\mu\nu}\delta^{ab} + 2\delta^a_\mu \delta^b_\nu - 2\delta^b_\mu \delta^a_\nu . \tag{2.4}$$

From (2.3) and (2.4) it is possible to deduce $T^{a,b}_{\mu\nu}$ for any spin thanks to the composition law

$$T^{ab,cd}_{\mu\nu} = T^{a,c}_{\mu\nu}\delta^{bd} + \delta^{ac} T^{b,d}_{\mu\nu} - \delta_{\mu\nu}\delta^{ac}\delta^{bd} . \tag{2.5}$$

As a consequence, the T matrix for $s = n + \frac{1}{2}$ can be deduced from that

([6]) M. Fierz and W. Pauli: *Proc. Roy. Soc.*, A **173**, 211 (1939).

([7]) C. Fronsdal: *Suppl. Nuovo Cimento*, **9**, 416 (1958).

([8]) E. M. Corson: *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (London-Glasgow, 1953), chap. I.

for $s = n =$ integer by the substitution $\delta_{\mu\nu} \to \gamma_\mu \gamma_\nu$. *E.g.*, for $s = \frac{3}{2}$

$$T^{a,b}_{\mu\nu} = \gamma_\mu \gamma_\nu \delta^{ab} + 2\delta^a_\mu \delta^b_\nu - 2\delta^b_\mu \delta^a_\nu . \tag{2.6}$$

In what follows the T matrix defined by (2.1), (2.2), will be considered as the natural basic link between the particle and the electromagnetic field.

3. – The equation of motion.

If we forget for a while the constraint equation (1.2) we can immediately find a gauge invariant interaction containing $T_{\mu\nu}$. In fact, the Klein-Gordon equation may be written

$$(- T_{\mu\nu} p_\mu p_\nu + m^2)\Psi = 0$$

and carrying out the usual substitution $p_\mu \to p_\mu - eA_\mu$, we obtain

$$(p^2 - m^2)\Psi = I\Psi , \tag{3.1}$$

where the interaction operator is

$$I = T_{\mu\nu}(ep_\mu A_\nu + eA_\mu p_\nu + e^2 A_\mu A_\nu) . \tag{3.2}$$

Of course, what we want is an equation in which due account is taken of the constraint (1.2). As a first step in that direction let us only accept the solutions of (3.1) for which the incoming field ([9]) satisfies (1.2). The solution itself may be divided into two parts; the projection

$$\Psi_P = P\Psi \tag{3.3}$$

and the complementary field

$$\Psi_Q = (1 - P)\Psi = Q\Psi . \tag{3.4}$$

Because of the condition on the incoming field, Ψ_Q is « weaker » than Ψ and goes to zero with the coupling constant. For this reason we will refer to Ψ_P as the « strong » part and to Ψ_Q as the « weak » part of Ψ ([10]).

Obviously, the strong part satisfies the constraint (1.2) and having been derived from a solution of a gauge invariant equation of motion, we may adopt the point of view that it completely represents the particle interacting with

([9]) C. N. YANG and D. FELDMAN: *Phys. Rev.*, **79**, 972 (1950).

([10]) Note that these definitions are covariant. For the free field case $\Psi_Q \equiv 0$.

the electromagnetic field. The weak field (3.4) is merely needed to maintain the gauge invariance of (3.1) and it is void of physical meaning. It will be shown in Section 6 that for $s = \frac{1}{2}$ this viewpoint yields the same result as the Dirac theory.

4. – Alternative approach.

The assertion that the weak part of Ψ has no physical meaning raises the question: Is it possible to get rid of Ψ_Q?

The problem we are confronted with is analogous to that of the construction of a Pauli equation equivalent to a Dirac one. In fact, with (3.3) and (3.4), the equation (3.1) splits into two coupled equations

$$(p^2 - m^2)\Psi_P = P \cdot I \cdot (\Psi_P + \Psi_Q) , \tag{4.1}$$

$$(p^2 - m^2)\Psi_Q = Q \cdot I \cdot (\Psi_P + \Psi_Q) . \tag{4.2}$$

And the problem is now that of the elimination of the weak part so as to obtain an equation of motion for Ψ_P alone.

Let us repeat that we are looking for a gauge invariant description of the interaction, compatible with the constraint. So far we have tried to obtain the wave function of the particle starting with the evidently gauge invariant equation (3.1) and then projecting the solutions (cf. (3.3)) so as to obtain a field satisfying (1.2). Let us now directly start with an equation which is evidently compatible with the constraint:

$$(p^2 - m^2)\Psi = P \cdot I' \Psi . \tag{4.3}$$

This is the sort of equation we get from a Lagrangian if we take into account the constraint (1.2) [1]. (4.3) has also the form of the equation we obtain for Ψ_P if we formally eliminate Ψ_Q in (4.1).

The gauge invariance condition must now determine the interaction operator I'. We will use the perturbation expansion as a tool for this determination. Firstly, we write

$$I' = I^{(1)} + I^{(2)} + \dots , \tag{4.4}$$

where $I^{(n)}$ contains n-times the electromagnetic potential.

For $I^{(1)}$ we choose the basic interaction

$$I^{(1)} = e \cdot T_{\mu\nu}(p_\mu A_\nu + A_\mu p_\nu) . \tag{4.5}$$

It is easy to see that with the choice (4.5) the equation (4.3) is gauge invariant in first order.

To obtain $I^{(2)}$ we turn our attention to the second approximation. We observe that the second order S-matrix for the equation (4.3) is proportional to

$$(4.6) \qquad I^{(2)}(\varepsilon^0, \varepsilon^1) + I^{(1)}(\varepsilon^1)\frac{P(p)}{p^2 - m^2} I^{(1)}(\varepsilon^0) + I^{(1)'}(\varepsilon^0)\frac{P(p')}{p'^2 - m^2} I^{(1)'}(\varepsilon^1) .$$

where, as usual

$$p_\mu = p^0_\mu + k^0_\mu = p^1_\mu + k^1_\mu ,$$
$$p'_\mu = p^0_\mu - k^1_\mu = p^1_\mu - k^0_\mu ,$$

p^0_μ (k^0_μ) being the initial momentum of the particle (photon) and p^1_μ (k^1_μ) being the final one. ε^0 and ε^1 are the polarization vectors of the initial and final photons.

Now, gauge invariance imposes the following condition: (4.6) must be identically zero when any one of the polarization vectors is substituted for the corresponding impulse vector of the photon. *I.e.*

$$(4.7) \qquad I^{(2)}(\varepsilon^0, k^1) + I^{(1)}(k^1)\frac{P(p)}{p^2 - m^2} I^{(1)}(\varepsilon^0) + I^{(1)'}(\varepsilon^0)\frac{P(p')}{p'^2 - m^2} I^{(1)'}(k^1) \equiv 0 .$$

But,

$$k^1_\mu = p_\mu - p^1_\mu = p^0_\mu - p'_\mu$$

so that

$$I^{(1)}(k^1) = T_{\mu\nu}(p^1_\mu k^1_\nu + k^1_\mu p_\nu) = T_{\mu\nu}(-p^1_\mu p^1_\nu + p_\mu p_\nu) = m^2 - p^2 ,$$
$$I^{(1)'}(k^1) = T_{\mu\nu}(p'_\mu k^1_\mu + k^1_\mu p^0_\nu) = T_{\mu\nu}(-p'_\mu p'_\nu + p^0_\mu p^0_\nu) = p'^2 - m^2 .$$

Therefore, (4.7) is

$$(4.8) \qquad I^{(2)}(\varepsilon^0, k^1) - e\cdot P(p)\cdot I^{(1)}(\varepsilon^0) + e\cdot I^{(1)'}(\varepsilon^0)\cdot P(p') \equiv 0 .$$

This identity will now be viewed as a definition of $I^{(2)}(\varepsilon^0, k^1)$. *I.e.*

$$I^{(2)}(\varepsilon^0, k^1) \equiv e\cdot P(p)\cdot I^{(1)}(\varepsilon^0) - e\cdot I^{(1)'}(\varepsilon^0)\cdot P(p')$$

or more explicitly

$$(4.9) \qquad I^{(2)}(\varepsilon^0, k^1) \equiv e^2\cdot P(p)\cdot T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) - e^2 T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p'_\nu)\cdot P(p') .$$

To find out $I^{(2)}(\varepsilon^0, \varepsilon^1)$ we need the explicit presence of k^1 in the right hand side of (4.9). We note that $P(p) = P(p^1 + k^1)$ has a part independent of k^1,

namely $P(p^1)$. This part, acting on the final state is equivalent to one. Then, to separate this part we introduce again the complementary projection operator $Q = 1 - P$. Eq. (4.9) is equivalent

$$(4.10) \qquad I^{(2)}(\varepsilon^0, k^1) = e^2 \cdot T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) - e^2 \cdot T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^1_\nu) - \\ - e^2 \cdot Q(p) \cdot T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) + e^2 \cdot T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^1_\nu) \cdot Q(p') \,.$$

The first and second terms in the right hand side of (4.10) can be simplified

$$T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) - T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p'_\nu) = T_{\mu\nu}(k^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu k^1_\nu) = 2\varepsilon^0 \cdot k^1 \,.$$

Therefore

$$(4.11) \qquad I^{(2)}(\varepsilon^0, k^1) = e^2\{2\varepsilon^0 \cdot k^1 - Q(p) \cdot T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) + T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p'_\nu) \cdot Q(p')\} \,.$$

Now, $I^{(2)}(\varepsilon^0, \varepsilon^1)$ can be obtained from (4.11) replacing in each term one of the k^1 by ε^1, and taking into account the symmetry between both photons. Note that the first term in the right hand side of (4.11) gives the usual A^2 term.

Actually, the oulined method does not determine $I^{(2)}(\varepsilon^0, \varepsilon^1)$ itself but rather its matrix elements taken between free particle states. Nevertheless this is all what we need for the second order approximation. The method can be similarly applied to the third and higher order terms. In each order of the perturbation expansion the gauge invariance condition allows one to write an expression similar to (4.7), which is then used to find out the corresponding part of the interaction operator (4.4). It can also be seen that, because of its definition, each one of the higher order interaction terms brings a contribution to the S-matrix which has the same degree of divergence (or convergence) as that of $I^{(1)}$. The contribution of the latter is easily seen to be independent of the spin because the projection operator does not contribute to the degree of divergence. Therefore, the theory is renormalizable for any spin value if it is so for a particular one. It is known that for spin zero the theory is renormalizable. It follows then the renormalizability for any spin.

5. – Examples.

To see how (4.11) yields $I^{(2)}(\varepsilon^0, \varepsilon^1)$ in actual cases, let us treat two examples.

a) *Spin* $\frac{1}{2}$. – From (1.3)

$$Q(p) = \frac{1}{2}\left(1 + i\frac{\gamma \cdot p}{p}\right), \qquad i\gamma \cdot p Q(p) = p \cdot Q(p) \,.$$

From (2.3)

$$Q(p)\cdot T_{\mu\nu}(p_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) = Q(p)\cdot(\gamma\cdot p\,\gamma\cdot\varepsilon^0 + \gamma\cdot\varepsilon^0\,\gamma\cdot p^0) =$$
$$= -\,i\cdot Q(p)\cdot(p\,\gamma\cdot\varepsilon^0 - \gamma\cdot\varepsilon^0 m) = -\,i(p-m)\cdot Q(p)\,\gamma\cdot\varepsilon^0\,.$$

Also

$$T_{\mu\nu}(p^1_\mu \varepsilon^0_\nu + \varepsilon^0_\mu p^1_\nu)\cdot Q(p') = -\,i(p'-m)\gamma\cdot\varepsilon^0 Q(p')\,.$$

To extract a $\gamma\cdot k^1$ factor from Q, we observe that

$$i\gamma\cdot k^1\cdot Q(p) = (i\gamma\cdot p - i\gamma\cdot p') = (p+m)\,Q(p)$$

and

$$Q(p')\cdot i\gamma\cdot k^1 = Q(p')(i\gamma\cdot p^0 - i\gamma\cdot p') = -\,(p'+m)\,Q(p')\,.$$

So that, from (4.11)

$$I^{(2)}(\varepsilon^0, k^1) = e^2\left\{2\varepsilon^0\cdot k^1 - \gamma\cdot k^1\frac{p-m}{p+m}\,Q(p)\,\gamma\cdot\varepsilon^0 - \gamma\cdot\varepsilon^0\frac{p'-m}{p'+m}\,Q(p')\gamma\cdot k^1\right\}.$$

Therefore

$$(5.1)\qquad I^{(2)}(\varepsilon^0, \varepsilon^1) = e^2\left\{2\varepsilon^0\cdot\varepsilon^1 - \gamma\cdot\varepsilon^1\frac{p-m}{p+m}\,Q(p)\,\gamma\cdot\varepsilon^0 - \gamma\cdot\varepsilon^0\frac{p'-m}{p'+m}\,Q(p')\gamma\cdot\varepsilon^1\right\}.$$

b) *Spin* 1. – From (1.4)

$$Q^{\alpha\beta}(p) = -\frac{p^\alpha p^\beta}{p^2}\,.$$

From (2.4)

$$Q^{\alpha\beta}(p)\cdot T^{\beta\gamma}_{\mu\nu}(p_\mu\varepsilon^0_\nu + \varepsilon^0_\mu p^0_\nu) = \frac{p^\alpha}{p^2}\cdot 2(p^2 + p\cdot p^0)\varepsilon^{0\gamma} = k^{1\alpha}\left(1 - \frac{m^2}{p^2}\right)\varepsilon^{0\gamma}\,.$$

Also

$$T^{\alpha\beta}_{\mu\nu}(p^1_\mu\varepsilon^0_\nu + \varepsilon^0_\mu p'_\nu)\cdot Q^{\beta\gamma}(p') = -\,\varepsilon^{0\alpha}\left(1 - \frac{m^2}{p'^2}\right)k^{1\gamma}\,.$$

Therefore

$$I^{(2)}(\varepsilon^0, k^1) = e^2\left\{2\varepsilon^0\cdot k^1\,\delta^{\alpha\beta} + k^{1\alpha}\left(1 - \frac{m^2}{p^2}\right)\varepsilon^{0\beta} + \varepsilon^{0\alpha}\left(1 - \frac{m^2}{p'^2}\right)k^{1\beta}\right\},$$

$$(5.2)\qquad I^{(2)}(\varepsilon^0, \varepsilon^1) = e^2\left\{2\varepsilon^0\cdot\varepsilon^1\,\delta^{\alpha\beta} + \varepsilon^{1\alpha}\left(1 - \frac{m^2}{p^2}\right)\varepsilon^{0\beta} + \varepsilon^{0\alpha}\left(1 - \frac{m^2}{p'^2}\right)\varepsilon^{1\beta}\right\}.$$

It is easy to realize that (5.1) and (5.2) have a common form, namely

$$(5.3)\qquad I^{(2)}(\varepsilon^0,\varepsilon^1)=e^2\Big\{2\varepsilon^0\cdot\varepsilon^1+T_{\mu\nu}(p^1_\mu\varepsilon^1_\nu+\varepsilon^1_\mu p_\nu)\frac{Q(p)}{p^2-m^2}\cdot$$

$$\cdot T_{\varrho\sigma}(p_\varrho\varepsilon^0_\sigma+\varepsilon^0_\varrho p^0_\sigma)+T_{\mu\nu}(p^1_\mu\varepsilon^0_\nu+\varepsilon^0_\mu p'_\nu)\frac{Q(p')}{p'^2-m^2}\cdot T_{\varrho\sigma}(p'_\varrho\varepsilon^1_\sigma+\varepsilon^1_\varrho p^0_\sigma)\Big\}\,.$$

Where, due to the presence of Q, there is no pole for $p^2\to m^2$. (Actually $I^{(2)}\to 2e^2\varepsilon^0\cdot\varepsilon^1$).

When we substitute (5.3) in (4.6), we find out that P and Q both disappear in the final result, which turns out to be

$$(5.4)\qquad 2e^2\varepsilon^0\cdot\varepsilon^1+e^2T_{\mu\nu}(p^1_\mu\varepsilon^1_\nu+\varepsilon^1_\mu p_\nu)\frac{1}{p^2-m^2}\cdot T_{\varrho\sigma}(p_\varrho\varepsilon^0_\sigma+\varepsilon^0_\varrho p^0_\sigma)+$$

$$+e^2T_{\mu\nu}(p^1_\mu\varepsilon^0_\nu+\varepsilon^0_\mu p'_\nu)\frac{1}{p'^2-m^2}\cdot T_{\varrho\sigma}(p'_\varrho\varepsilon^1_\sigma+\varepsilon^1_\varrho p_\sigma)$$

In harmony with the viewpoint expressed in Section 3.

6. – Spin $\frac{1}{2}$. Equivalence with Dirac's theory.

For $s=\frac{1}{2}$, the first order part of (3.2), *i.e.* the basic interaction (4.5) is

$$I^{(1)}=e\cdot(\gamma\cdot p'\,\gamma\cdot\varepsilon+\gamma\cdot\varepsilon\,\gamma\cdot p)\,;\qquad p'=p+k\,.$$

The matrix element of $I^{(1)}$, taken between any two free particle states, is

$$\langle f|I^{(1)}|i\rangle=e\cdot\langle f|(\gamma\cdot p'\,\gamma\cdot\varepsilon+\gamma\cdot\varepsilon\,\gamma\cdot p\,|i\rangle=ie\cdot\langle f|m\gamma\cdot\varepsilon+\gamma\cdot\varepsilon\,m)\,|i\rangle\,,$$

$$(6.1)\qquad \langle f|I^{(1)}|i\rangle=2iem\langle f|\gamma\cdot\varepsilon\,|i\rangle\,.$$

The second order S-matrix element for the equation (3.11) (see also (5.4)) is proportional to

$$e^2\langle f|\Big\{(\gamma\cdot p^1\gamma\cdot\varepsilon^1+\gamma\cdot\varepsilon^1\gamma\cdot p)\frac{1}{p^2-m^2}(\gamma\cdot p\gamma\cdot\varepsilon^0+\gamma\cdot\varepsilon^0\gamma\cdot p^0)+$$

$$+(\gamma\cdot p^1\gamma\cdot\varepsilon^0+\gamma\cdot\varepsilon^0\gamma\cdot p')\frac{1}{p'^2-m^2}(\gamma\cdot p'\gamma\cdot\varepsilon^1+\gamma\cdot\varepsilon^1\gamma\cdot p^0)+2\varepsilon^0\cdot\varepsilon^1\Big\}|i\rangle=$$

$$=e^2\langle f|\Big\{\gamma\cdot\varepsilon^1\frac{(m-i\gamma\cdot p)(i\gamma\cdot p-m)}{p^2-m^2}\gamma\cdot\varepsilon^0+\gamma\cdot\varepsilon^0\frac{(m-i\gamma\cdot p')(i\gamma\cdot p'-m)}{p'^2-m^2}\cdot\gamma\cdot\varepsilon^1+$$

$$+\gamma\cdot\varepsilon^0\gamma\cdot\varepsilon^1+\gamma\cdot\varepsilon^1\gamma\cdot\varepsilon^0\Big\}|i\rangle=-e^2\langle f|\Big\{\gamma\cdot\varepsilon^1\frac{m^2+p^2-2mi\gamma\cdot p}{p^2-m^2}\gamma\cdot\varepsilon^0-\gamma\cdot\varepsilon^1\gamma\cdot\varepsilon^0+$$

$$+\gamma\cdot\varepsilon^0\frac{m^2+p'^2-2mi\gamma\cdot p'}{p'^2-m^2}\gamma\cdot\varepsilon^1-\gamma\cdot\varepsilon^0\gamma\cdot\varepsilon^1\Big\}|i\rangle\,,$$

$$(6.2) \qquad S_{fi}^{(2)} \cong -2me\langle f|\left\{\gamma\cdot\varepsilon^1\frac{m-i\gamma\cdot p}{p^2-m^2}\gamma\cdot\varepsilon^0+\gamma\cdot\varepsilon^0\frac{m-i\gamma\cdot p'}{p'^2-m^2}\gamma\cdot\varepsilon^1\right\}|i\rangle\,.$$

(6.1) and (6.2) coincide with the matrix elements obtained with Dirac's theory. By a similar procedure it may be shown that the coincidence holds also in higher order. It is curious that here the factor $(m-i\gamma\cdot p)$ comes from the interaction, not from the propagator.

7. – Magnetic moments and gyromagnetic factor.

The value of the magnetic moment of the elementary particles may be obtained from the fundamental interaction (4.5)

$$I^{(1)} = e\cdot T_{\mu\nu}(p_\mu A_\nu + A_\mu p_\nu)$$

and the expression (2.2) for the transformation matrix. The substitution of (2.2) in (4.5) yields

$$I^{(1)} = e\cdot(\delta_{\mu\nu}+2iS_{\mu\nu})(p_\mu A_\nu + A_\mu p_\nu)\,,$$

$$I^{(1)} = e\cdot(p\cdot A + A\cdot p) + 2ie\cdot S_{\mu\nu}(p_\mu A_\nu + A_\mu p_\nu)\,.$$

By replacing $p_\mu \to -i\partial_\mu$ into the spin part of the latter formula we can obtain

$$I^{(1)} = 2e\,A\cdot p + 2ie\cdot S_{\mu\nu}(-i\,\partial_\mu A_\nu)$$

and taking into account the antisymmetry of the spin matrix

$$(7.1) \qquad I^{(1)} = 2e\,A\cdot p + eS_{\mu\nu}F_{\mu\nu}\,,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

(7.1) is an explicit separation of $I^{(1)}$ into two parts, the four-current interaction and the spin interaction. The latter contains the magnetic moment interaction.

The non-relativistic limit of (7.1) tells us that an elementary particle of spin s has a magnetic moment of magnitude $2s$ Bohr magnetons. In other words, the magnetic moment is proportional to the spin of the particle and the gyromagnetic factor has the value 2 for any spin.

* * *

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RIASSUNTO (*)

Si tenta di costruire una teoria secondo la quale tutte le condizioni supplementari siano considerate come equazioni di costrizione ed implicanti che solo $2s+1$ (s=spin) delle sue componenti siano realmente indipendenti. Nel caso dei campi liberi di cui si è già trattato, tutte le particelle elementari obbediscono all'equazione del moto di Klein-Gordon. Per introdurre in maniera generale il campo elettromagnetico, si considera l'interazione fondamentale che coinvolge la matrice di trasformazione della funzione d'onda. Si descrive un processo per mezzo del quale si può costruire un'equazione invariante rispetto al « gauge » del moto. La teoria è rinormalizzabile; essa descrive delle particelle che hanno il fattore giromagnetico normale 2, per qualsiasi valore dello spin.

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On the Phase Factors in Inversions (*).

G. FEINBERG (**)

Brookhaven National Laboratory - Upton, N.Y.

S. WEINBERG (***)

Columbia University - New York, N.Y.

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Summary. — The phase factors which can appear in the definition of the inversions, C, P, T and their products are discussed. It is shown that because of the existence of « physically equivalent » Hamiltonians, the phases in C, CP, T and TP for complex fields are unmeasurable. For the remaining inversions, it is possible to construct interactions which require more general phases for complex fields than the usual ± 1, $\pm i$, when and only when the theory contains certain discrete multiplicative symmetries. Examples of such interactions are given.

1. – Introduction.

The transformations of the field operators of a quantized field theory, which are generated when the state vectors undergo inversion operations such as space reflection (P), charge conjugation (C) and time reversal (T) are not entirely determined *a priori*. In particular, when these operators act on the complex fields which represent particles different from their antiparticles, there is the possibility of introducing arbitrary complex numbers of modulus 1 (phase factors) into the action of the inversion on such fields, while still maintaining their unitary character, the invariance of the free field Lagrangian and the

(*) Supported in part by the U.S. Atomic Energy Commission.

(**) Now at the Department of Physics, Columbia University, New York, N.Y.

(***) Now at the Lawrence Radiation Laboratory, Berkeley, Cal.

required transformation properties of free field observables such as currents, momenta, etc. The existence of this formal possibility has long been realized, but there has been some dispute as to its physical content, particularly in the case of space reflection. The purpose of this paper is to discuss the restrictions on the phase factors which may exist in field theories and the physical content of such restrictions. A consequence of our discussion is a specification of which phases are purely conventional and which are fixed by a particular theory, and hence may, at least in principle, be determined by experiment. The main results are contained in the following two statements:

1) In a theory invariant under charge conjugation the phase factors which occur in the transformation of complex fields under $\boldsymbol{C}$ can always be chosen to have any value for any such field without altering the physical content of a theory, and so are not observable quantities.

A similar result holds for the phase factors occurring in time reversal. However, the product of the phases in $\boldsymbol{C}$ and $\boldsymbol{T}$ cannot be chosen arbitrarily, and is related to the phase factor in $\boldsymbol{P}$ for each field, in a way to be discussed, provided that the $\boldsymbol{TCP}$ theorem is satisfied.

2) In a theory invariant under space reflection it is not in general possible to restrict the phase factor in $\boldsymbol{P}$ to the « usual » values (± 1 for boson fields, and ± 1, $\pm i$ for fermion fields). That is, it is possible to construct theories which are invariant under space reflection operations involving more general phase factor than these, but which are not invariant with any of the usual phase factors.

Such theories are characterized by the existence of new multiplicative symmetry operations which are not part of continuous gauge groups. No examples are known for any of the commonly accepted interactions of the known elementary particles. While the formal theory of inversions can be carried through most easily by working with the fields and the corresponding phase factors, the physically interesting quantities are the phase factors occurring in the transformation of states, the so-called intrinsic parities. It will be shown that intrinsic parities may be compared only for states which have the same transformation under all multiplicative symmetry operations.

In the second section of this paper, we will define the operations $\boldsymbol{P}$, $\boldsymbol{C}$, $\boldsymbol{T}$ as well as certain other operators and discuss several of their properties. In the third section, we discuss the ambiguity in inversions due to the existence of many physically equivalent Hamiltonians, and justify the first statement above. In the fourth section we discuss the consequence for inversions of the existence of multiplicative symmetries. In the final section we will prove certain relations among products of the inversions.

2. – Inversion operators and multiplicative operators.

In this paper, we consider only theories in which the degeneracy of the 1 particle states is completely specified by the spins and the particle-antiparticle character. This means that the transformations representing the inversions can only take the field operator into itself or into its Hermitian conjugate, rather than permuting fields which refer to different particles. Theories in which there is additional degeneracy and in which the space reflection and time reversal operators are more general have been considered by WIGNER and by MICHEL and WIGHTMAN [1].

The form of the inversion operators is chosen to make the observables transform in accordance with the classical interpretation of these operations. For example, we make the *a priori* requirement that space reflection should not change particles to antiparticles but should change the sign of momentum, while charge conjugation should change particle to antiparticle without changing momentum. The use of the term parity for an operation which does not satisfy the first criterion appears unwarranted on the basis of the classical concept of space reflection.

Let $\varphi_m(X)$ be the operators for spinless boson fields, $\varphi_{\mu m}(X)$ the operators for spin 1 fields and $\psi_m(X)$ the operators for spin $\frac{1}{2}$ fields. The three inversions are defined by the following equations:

$$(1)\qquad \begin{cases} \boldsymbol{C}\varphi_m(X)\boldsymbol{C}^{-1} = n_m^c \varphi_m^\dagger(X)\,; & \boldsymbol{C}\varphi_m^\dagger(X)\boldsymbol{C}^{-1} = n_m^{*c}\varphi_m(X)\,, \\ \boldsymbol{C}\varphi_{\mu m}(X)\boldsymbol{C}^{-1} = n_m^c \varphi_{\mu m}^\dagger(X)\,; & \boldsymbol{C}\varphi_{\mu m}^\dagger(X)\boldsymbol{C}^{-1} = n_m^{*c}\varphi_{\mu m}(X)\,, \\ \boldsymbol{C}\psi_m(X)\boldsymbol{C}^{-1} = n_m^c\, C\,\overline{\psi}_m^T(X)\,; & \boldsymbol{C}\overline{\psi}_m(X)\boldsymbol{C}^{-1} = -\,n_m^{*c}\psi_m^T(X)C^+. \end{cases}$$

Here C is the usual charge conjugation matrix satisfying

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu\,.$$

We use hermitian γ_μ satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\,, \qquad (\mu = 1,\ 2,\ 3,\ 4,\ \text{always})\,.$$

The superscript T on a matrix or a field indicates transposition in the spin space. The dagger operation means hermitian conjugation in Hilbert space,

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[1] See L. MICHEL and A. WIGHTMAN: *Lecture notes at Princeton University* (unpublished).

together with transposition of spinor indices. The operator $\boldsymbol{C}$ is linear and unitary. For spin 1 fields we shall let μ take the values 0, 1, 2, 3, where $\varphi_0 \equiv -i\varphi_4$.

$$(2)\quad \begin{cases} \boldsymbol{P}\varphi_m(X)\boldsymbol{P}^{-1} = n_m^P \varphi_m(\varrho X); & \boldsymbol{P}\varphi_m^\dagger(X)\boldsymbol{P}^{-1} = n_m^{P*}\varphi_m^\dagger(\varrho X), \\ \boldsymbol{P}\varphi_{\mu m}(X)\boldsymbol{P}^{-1} = n_m^P \varrho\varphi_{\mu m}(\varrho X); & \boldsymbol{P}\varphi_{\mu m}^\dagger(X)\boldsymbol{P}^{-1} = n_m^{P*}\varrho\varphi_{\mu m}^\dagger(\varrho X), \\ \boldsymbol{P}\psi_m(X)\boldsymbol{P}^{-1} = in_m^P\gamma_4\psi_m(\varrho X); & \boldsymbol{P}\bar{\psi}_{\mu m}(X)\boldsymbol{P}^{-1} = -in_m^{P*}\bar{\psi}_m(\varrho X)\gamma_4. \end{cases}$$

Here ϱX represents the transformed coordinates, *i.e.* $\varrho(\boldsymbol{X}, t) = (-\boldsymbol{X}, t)$. Similarly $\varrho\varphi_{im} = -\varphi_{im}$ for the space components of a vector field, and $\varrho\varphi_{4m} = \varphi_{4m}$ for the time component of a vector field. $\boldsymbol{P}$ is also a linear unitary operator.

$$(3)\quad \begin{cases} \boldsymbol{T}\varphi_m(X)\boldsymbol{T}^{-1} = n_m^T \varphi_m(\tau X); & \boldsymbol{T}\varphi_m^\dagger(X)\boldsymbol{T}^{-1} = n_m^{T*}\varphi_m^\dagger(\tau X), \\ \boldsymbol{T}\varphi_{\mu m}(X)\boldsymbol{T}^{-1} = n_m^T \tau\varphi_m(\tau X); & \boldsymbol{T}\varphi_{\mu m}^\dagger(X)\boldsymbol{T}^{-1} = n_m^T \tau\varphi_{\mu m}^\dagger(\tau X), \\ \boldsymbol{T}\psi_m(X)\boldsymbol{T}^{-1} = n_m^T C^{-1}\gamma_5\psi_m(\tau X); & \boldsymbol{T}\bar{\psi}_m(X)\boldsymbol{T}^{-1} = n_m^{T*}\bar{\psi}_m(\tau X)\gamma_5 C. \end{cases}$$

$\boldsymbol{T}$ is an antiunitary operator, *i.e.* $\boldsymbol{T}\lambda\boldsymbol{T}^\dagger = \lambda^*$ for any c-number. τX again represents the transformed coordinate, *i.e.* $\tau(X, t) = (X, -t)$.

The conditions that $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{T}$ should be unitary (antiunitary) and that the free field Lagrangian is invariant restrict n_m^C, n_m^P, n_m^T to be phase factors, *i.e.* $|n_m^{C,P,T}| = 1$ for any field.

For the « real » fields which represent particles identical to their antiparticles, one has the additional conditions

$$(4)\quad \begin{cases} \varphi_r = \varphi_r^\dagger = \varphi_r^C, \\ \varphi_\mu = \varphi_{\mu r}^\dagger = \varphi_{\mu r}^C, \\ \psi_r = C\bar{\psi}_r^T = \psi_r^C. \end{cases}$$

These conditions just insure the identity of the particle and antiparticle states. By substituting these conditions into equations (1), (2), (3), it is easily seen that n_m^C, n_m^P, n_m^T must all be real for such fields, and thus are restricted to the values ± 1.

These are the only conditions on the phase factors which are imposed by the general requirements on the inversions, which involve the free field observables. A theory will be said to be invariant under one of the inversions whenever the phase factors n_m for the fields appearing in the theory can be chosen so that the inversion as defined with these phases commutes with the total Hamiltonian. Furthermore, if more than one choice of phase lead to inversion operators which commute with H, then any of the distinct operators

which are defined by the different choices of phase can be chosen to represent the inversion.

The physical content of this definition can be illustrated for the case of space reflection. A theory is invariant under space reflection if no experiment enables one to distinguish between left and right. This will be the case if and only if in all transitions from an initial state with a definite orbital parity to final states containing a specified set of particles, the orbital parities of the final states are always the same. It is easy to see that this condition is satisfied whenever there is invariance in the sense we have used above. Furthermore, the assignment of intrinsic parities to particles by the use of the phase factors n_m^P is from this point of wiev a way of keeping track of the fact that when transitions take place between certain particles, the orbital parities change in a certain way. Any choice of phase consistent with the unitarity requirement and the invariance of the Hamiltonian is to be admitted, whenever such a choice is required to summarize a physical consequence of the theory. In the fourth section we will see an example of how this possibility of arbitrary phases is actually necessary for space reflection. We have stated these rather trivial points here at such lenght because they are in conflict with views which have sometimes been advanced. We believe that any requirement to be made on the inversion operators beyond the ones we have stated must involve additional physical assumptions which should always be emphasized clearly.

In the subsequant discussion a class of unitary operators to be referred to as multiplicative operators will be met with frequently. A multiplicative operator U is defined by

$$(5) \qquad U\varphi_m(X)U^{-1} = n_m(U)\varphi_m(X); \qquad U\varphi_m^\dagger U^{-1} = n_m^*(U)\varphi_m^\dagger,$$

where the φ_m are boson or fermion fields, and the $n_m(U)$ are phase factors. Each U is prescribed by giving the quantities n_m for all the fields, which in general are different for the different fields. Since U is assumed unitary the n_m again satisfy $|n_m| = 1$. Also for real fields $n_r = \pm 1$.

Several properties of multiplicative operators follow from the definition.

a) The product of any number of multiplicative operators is a multiplicative operator.

b) The inverse of a multiplicative operator is a multiplicative operator, whose phases satisfy $n_m(U^{-1}) = n_m^*(U)$.

c) Define $\sqrt{U}$, a square root of a multiplicative operator U, by $\sqrt{U}\varphi_m\sqrt{U^\dagger} = \sqrt{n_m(U)}\varphi_m$.
Then $\sqrt{U}$ is also a multiplicative operator satisfying $\sqrt{U}\sqrt{U} = U$. There

are a number of such square roots, corresponding to the choice of sign of $\sqrt{n_m(U)}$ for each m.

d) Any two multiplicative operators commute.

e) Let U be any multiplicative operator. Then

$$\text{(6)} \qquad \begin{cases} \boldsymbol{U}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{U}\,, \\ \boldsymbol{U}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{U}^{\dagger}\,, \\ \boldsymbol{U}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{U}^{\dagger}\,. \end{cases}$$

We prove the results of eq. (6) for a spinless boson, the proof being trivially generalized to other cases

$$\boldsymbol{U}\boldsymbol{P}\varphi_m(X)\boldsymbol{P}^{-1}\boldsymbol{U}^{-1} = \boldsymbol{U}n_m^{P}\varphi_m(\varrho X)\boldsymbol{U}^{-1} = n_m^{P}n_m(U)\varphi_m(\varrho X)\,,$$

$$\boldsymbol{P}\boldsymbol{U}\varphi_m(X)\boldsymbol{U}^{-1}\boldsymbol{P}^{-1} = \boldsymbol{P}n_m\varphi_m\ (X)\boldsymbol{P}^{-1} = n_m^{P}n_m(U)\varphi_m(\varrho X)\,,$$

so $$\boldsymbol{U}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{U}\,,$$

$$\boldsymbol{U}\boldsymbol{C}\varphi_m\ (X)\ \boldsymbol{C}^{-1}\boldsymbol{U}^{-1} = \boldsymbol{U}n_m^{\sigma}\varphi_m^{\dagger}(X)\boldsymbol{U}^{-1} = n_m^{\sigma}n_m^{*}\varphi_m^{\dagger}(X)\,,$$

$$\boldsymbol{C}\boldsymbol{U}^{\dagger}\varphi_m(X)\boldsymbol{U}^{\dagger-1}\boldsymbol{C}^{-1} = \boldsymbol{C}n_m^{*}\varphi_m\ (X)\boldsymbol{C}^{-1} = n_m^{\sigma}n_m^{*}\varphi_m^{\dagger}(X)\,,$$

so $$\boldsymbol{U}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{U}^{\dagger}\,,$$

$$\boldsymbol{U}\boldsymbol{T}\varphi_m(X)\boldsymbol{T}^{-1}\boldsymbol{U}^{-1} = \boldsymbol{U}n_m^{T}\varphi_m(\tau X)\boldsymbol{U}^{-1} = n_m^{T}n_m\varphi_m(\tau X)\,,$$

$$\boldsymbol{T}\boldsymbol{U}^{\dagger}\varphi_m(X)\boldsymbol{U}\boldsymbol{T}^{-1} = \boldsymbol{T}n_m^{*}\varphi_m\ (X)\,\boldsymbol{T}^{-1} = n_m^{T}n_m\varphi_m(\tau X)\,,$$

where the last eq. follows because $\boldsymbol{T}$ is antilinear;

so $$\boldsymbol{U}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{U}^{\dagger}.$$

If a multiplicative operator commutes with the total Hamiltonian, it will be called a multiplicative symmetry operator.

A well known class of multiplicative operators is given by the gauge transformations, U_λ defined by

$$U_\lambda\varphi_m U_\lambda^{-1} = \exp[iq_m\lambda]\varphi_m\,,$$

where λ is an arbitrary number which is the same for all fields φ_m, while q_m is a number, usually an integer, which varies from field to field and represents

the « charge » of the particle which is conserved if the gauge transformations commute with the Hamiltonian. Certain multiplicative operators occur naturally when discussing inversions. In particular, it may be seen from the physical requirement of how inversions act on states that the square of each inversion operator should be a multiplicative operator, as the double application of an inversion must take each state into a physically equivalent state. Furthermore, the « commutator » of the inversions I_1 and I_2 defined by $I_1 I_2 I_1^{-1} I_2^{-1}$ should also be a multiplicative operator.

It is easy to verify this with the definitions given for the inversions in (1), (2), (3). In particular

$$\boldsymbol{C}^2 = 1 , \tag{7a}$$

$$\boldsymbol{P}^2 = GF , \tag{7b}$$

$$\boldsymbol{T}^2 = F , \tag{7c}$$

Here 1 is the identity operator, and F and G are multiplicative operators defined by

$$G\varphi_m G^{-1} = (n_m^P)^2 \varphi_m , \tag{8}$$

for any field, fermion or boson

$$\begin{cases} F\varphi_m F^{-1} = \varphi_m , & \text{for boson fields} , \\ F\psi_m F^{-1} = -\psi_m , & \text{for fermion fields} . \end{cases} \tag{9}$$

F will be recognized as the operator $(-1)^{N_F}$ where N_F is the total number of fermions. It is also the operator of rotation through 360° about any axis, and so will commute with the Hamiltonian in any theory invariant under proper Lorentz transformations. Note that $F^2 = 1$.

It should be stressed that eqs. (7*a*) and (7*c*) are true for all choices of the phase factor n_m^C, n_m^T whereas by (8), the form of $\boldsymbol{P}^2$ depends on n_m^P.

If the inversions $P(T)$ commute with H for some particular theory, then so will their squares $FG(F)$, and the theory will at least contain these as multiplicative symmetry operators.

If a theory is invariant under all three inversions it will in general process an additional multiplicative symmetry E. To see this, consider the operator $\boldsymbol{CPT}$.

$$\begin{cases} \boldsymbol{CPT}\varphi_m(X)\ (\boldsymbol{CPT})^{-1} = n_m^P n_m^C n_m^T \varphi_m(\varrho\tau X) , \\ \boldsymbol{CPT}\varphi_{\mu m}(X)(\boldsymbol{CPT})^{-1} = n_m^P n_m^C n_m^T \varrho\tau\varphi_{\mu m}(\varrho\tau X) , \\ \boldsymbol{CPT}\psi_m(X)\ (\boldsymbol{CPT})^{-1} = -in_m^P n_m^C n_m^T \gamma_5^T \psi_m^{\dagger T}(\varrho\tau X) . \end{cases} \tag{10}$$

If $\boldsymbol{C}$, $\boldsymbol{P}$ and $\boldsymbol{T}$ each commute with H then so will this product, where the phases n^P, n^C, n^T are the same ones that make the separate inversions commute.

In the proof of the ***TCP*** theorem ([2]), it is shown that the product θ of the operations of strong reflections and hermitian conjugation, defined by:

$$11) \qquad \begin{cases} \theta\varphi_m(X)\ \theta^{-1} = \varphi_m(\varrho\tau X) \\ \theta\varphi_{\mu m}(X)\theta^{-1} = \varrho\tau\varphi_{\mu m}(\varrho\tau X) \\ \theta\psi_m(X)\ \theta^{-1} = -i\gamma_5^T\psi_m^{\dagger T}(\varrho\tau X)\,. \end{cases}$$

will always commute with H for a local, Lorentz invariant theory with the usual connection between spin and statistics. Comparing this with the definition of ***CPT***, we conclude that for theories where the TCP theorem is true, if $\boldsymbol{T}$, $\boldsymbol{C}$ and $\boldsymbol{P}$ separately commute with H, then the multiplicative operator E defined by

$$(12) \qquad E\varphi_m E^{-1} = n_m^P n_m^C n_m^T \varphi_m\,,$$

will commute with H where n_m^P, n_m^C, n_m^T are any phases for which the separate inversions commute with H .

3. – Ambiguities due to physically equivalent Hamiltonians.

It is been recognized by PAULI ([3]), and others, that the relation between observable quantities such as transition probabilities, and the interaction Hamiltonian, is in general not one to one, but rather one to many. In particular, there may be many different interaction Hamiltonians involving the same particles, which lead to identical transition probabilities between any two states.

A theorem expressing this possibility can be stated as follows ([3]): Let U be a unitary transformation which leaves invariant (up to a multiplicative phase factor) the initial and final states for some process. Then the two interaction Hamiltonians H and UHU^{-1} lead to the same transition probabilities for the process.

In particular, if a transformation U multiplies all free particle states (« in » states) by arbitrary phase factors (which may vary from state to state) then the physical consequences (as expressed by transition probabilities) of

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([2]) G. LÜDERS: *Ann. Phys.*, **2**, 1 (1957).

([3]) W. PAULI: *Nuovo Cimento*, **6**, 204 (1957). See also D. PURSEY: *Nuovo Cimento*, **6**, 266 (1957).

the theories in which H or UHU^{-1} are the interactions, are identical. Note that we are not simply expressing the unitary equivalence of the two theories, which is trivial. The transition probability is to be computed between the same states for each Hamiltonian, rather than between unitarily transformed states.

The multiplicative transformations U defined in the previous section are examples of such transformations. This is because the « in » states are defined as eigenstates of the free particle Hamiltonian, with quantum numbers given by the free particle observables. But the multiplicative operators commute with the free particle Hamiltonian and all the free particle observables. It therefore follows that for any « in » state $|\psi\rangle$, and any multiplicative operator U,

$$U|\psi\rangle = n_\psi(U)|\psi\rangle\,, \tag{13}$$

where $|n_\psi| = 1$ and in general depends on the state $|\psi\rangle$, as well as on U.

We will now demonstrate the above stated theorem for the operators U. Consider the two interaction Hamiltonians H and $H' = UHU^{-1}$. The S-matrices calculated from H and H' are clearly related by

$$S' = USU^{-1}\,.$$

Then if $|a\rangle$ and $|b\rangle$ are any two « in » states, it follows that

$$\langle a|S'|b\rangle = \langle a|USU^{-1}|b\rangle\,. \tag{14}$$

But by the above,

$$U^{-1}|a\rangle = n_a^*|a\rangle\,,$$

$$U^{-1}|b\rangle = n_b^*|b\rangle\,,$$

$$\langle a|S'|b\rangle = n_a n_b^*\langle a|S|b\rangle\,. \tag{15}$$

It follows from this that the two transition probabilities

$$\begin{cases} P_{ab} = |\langle a|S|b\rangle|^2\,, \\ P'_{ab} = |\langle a|S'|b\rangle|^2\,, \end{cases} \tag{16}$$

are equal for any states $|a\rangle$, $|b\rangle$, which is Pauli's theorem in this case. The Hamiltonians H and H' therefore describe the same physical system, and so can be used interchangeably without any change in the states. Two such Hamiltonians will be called equivalent.

This holds for states which contain fixed numbers of particles of each type.

States which are superpositions of such states, such as:

$$\sqrt{\tfrac{2}{3}}|n\pi^+\rangle + \exp[i\lambda]\sqrt{\tfrac{1}{3}}|p\pi^0\rangle$$

or

$$|K^0\rangle + \exp[i\lambda]|\bar{K}^0\rangle,$$

where λ is any real number, will not be transformed into themselves by arbitrary multiplicative transformations (4) but rather into states with different values of λ. This does not contradict our contention that the Hamiltonians H and UHU^{-1} are physically indistinguishable. This is because there are no experiments which directly determine the mixing phases λ for states like the above. To see this, we note that the matrix elements of all the free particle observables are independent of λ. Indeed, one can regard the introduction of the superposition states as merely a mathematical convenience. Only after a particular interaction Hamiltonian is chosen is it possible to distinguish between states with different mixing phases. But for a given choice of λ, the properties of the state will depend on which of the equivalent Hamiltonians is chosen, and so in the absence of an independent way of distinguishing between the different values of λ, this cannot be used to determine which of the equivalent Hamiltonian is correct.

For example, the eigenstates of total isotopic spin are superpositions of the above type. However, the isotopic spin operator can be specified only after choosing a particular set of coupling constants in the interaction Hamiltonian. When one transforms to an equivalent Hamiltonian, the isotopic spin operator, I, will also change to UIU^{-1}, unlike the free field observables. The U-transformed isotopic spin states will be eigenstates of the transformed isotopic spin operator. This implies that transition probabilities between states of definite isotopic spin are also unchanged by the multiplicative transformations. It may therefore be seen that the use of isotopic spin as an observable does not allow one to distinguish between equivalent Hamiltonians.

For comparison, instead of a multiplicative transformation consider an operator like P. The interactions H and PHP^{-1} are not equivalent unless P commutes with H, because some free particle observables, such as momentum, are not invariant under P, and therefore H and PHP^{-1}, give different transition probabilities for states of fixed momentum, which is an observable distinction. For example, if a Hamiltonian contains a spinor field always in the

(4) This was pointed out to the authors by Dr. G. C. Wick. We thank Dr. Wick for very helpful discussions of this and many other points in this paper.

form $(1+\gamma_5)\psi$, so that only left-handed particles interact, then PHP^{-1} will contain $(1-\gamma_5)\psi$ and here only right-handed particles interact.

In general U will not commute with H, and so H and H' will be different functions of the field operators. In particular, the phase factors of certain coupling constants may be altered after transforming with U. A corollary of this theorem is therefore that the absolute phase of the coupling constants for interactions which involve a complex field linearly is unobservable, since it can always be changed by transforming the Hamiltonian with a multiplicative operator without changing the physical content of the theory. Of course, the relative phase of coupling constants for several interactions involving the same fields may be measureable.

Suppose now that some Hamiltonian H is invariant under any inversion I, satisfying $IU = U^{\dagger}I$ (*e.g.*, ***C***, ***T***, ***CP***, ***PT***), with a particular set of phase factors n_m^I for the fields φ_m involved in H. This will mean that the coupling constants appearing in H will satisfy certain reality conditions, involving also the phase n_m^I. We can construct a Hamiltonian, equivalent in the sense defined previously, which is invariant under a new inversion in which the phase factors for all complex fields are $+1$ or any other number we choose, whereas the phase factors for real fields are unchanged.

To do this define a multiplicative operator U_I by

$$(17)\qquad \begin{cases} U_I \varphi_m U_I^{\dagger} = n_m(U_I)\varphi_m\,, \\ n_m(U_I) = \sqrt{n_m^I} \quad \text{for } \varphi_m \text{ any « complex » field}\,, \\ n_m(U_I) = \sqrt{1} \quad \text{for } \varphi_m \text{ any « real » field}\,. \end{cases}$$

Either square root may be chosen for each m. The definition of $n_m(U_I)$ for φ_m real is forced upon us by the condition that U_I be unitary, as discussed in Section **2**.

By hypothesis $[I, H] = 0$. Therefore

$$(18)\qquad [U_I I U_I^{\dagger},\ U_I H U_I^{\dagger}] = 0\,,$$

so that $I' = U_I I U_I^{\dagger}$ would be a suitable operator to represent the inversion with $U_I H U_I^{\dagger}$ as the Hamiltonian. But $U_I H U_I^{\dagger}$ is equivalent to H, and so we could adopt it as the Hamiltonian without changing the result of any experiment. It follows from eqs. (6) that the transformed inversion operator is

$$(19)\qquad I' = U_I I U_I^{-1} = U_I^2 I\,.$$

This operator will have all phase factors $+1$ for complex fields. To see this

in the case of $\boldsymbol{C}$ for example, one has for a spinless field

$$(20)\qquad \begin{cases} \boldsymbol{C}'\varphi\boldsymbol{C}'^{-1} = U_C^2\boldsymbol{C}\varphi\boldsymbol{C}^{-1}(U_C^{-1})^2 \\ \qquad\qquad = n^C U_C^2\varphi^\dagger(U_C^{-1})^2 \\ \qquad\qquad = n^C n^{C*}\varphi^\dagger, \\ \qquad\qquad = \varphi^\dagger. \end{cases}$$

so the phase is $+1$. Clearly, any other phase factor can be obtained for any complex field by choice of U_I. On the other hand, for real fields, $U_1^2 = 1$ so that $I' = I$ which means that the inversion is unchanged.

Since the results of experiments are invariant under transformations which change the phase factors in these inversions in an arbitrary way, the phase factors for complex fields must be unobservable, either absolutely or relative to each other. This is not so for real fields as we have seen. In particular the phases for the photon field $n^C = -1$, $n^P = 1$, $n^T = -1$, which make the electromagnetic interaction invariant, cannot be changed by such transformations, and thus can be determined by experiment.

While the phases appearing in $\boldsymbol{C}$, $\boldsymbol{T}$, $\boldsymbol{CP}$, $\boldsymbol{TP}$ are unobservable they cannot be simultaneously changed in an arbitrary way. This is because such inversions as $\boldsymbol{P}$, $\boldsymbol{CT}$ and $\boldsymbol{PCT}$ commute with multiplicative operators, and therefore the phases for these inversions will be unaltered by the transformation to an equivalent Hamiltonian. That is, for the equivalent Hamiltonian UHU^{-1}, the parity operator is

$$U\boldsymbol{P}U^{-1} = \boldsymbol{P},$$

and

$$U\boldsymbol{CT}U^{-1} = \boldsymbol{CT}.$$

These phase factors are then in principle measurable. The restrictions on measurements of such phases will be discussed in the next section.

The above results show that it is meaningless to ask for the relative n^C even for particles like Σ^0 and Λ^0 which can decay into each other by interactions which conserve C. Similarly, the relative n^T of the neutron and Λ^0 is not measurable even if the decay $\Lambda^0 \to \mathrm{n} + \pi^0$ conserves T. Experiments to measure these quantities therefore can not be devised.

4. – Ambiguities due to conservation laws.

In this Section we consider those inversion phases which are the same for all equivalent Hamiltonians. These include the phases in P, CT and PCT for any field, since these inversions commute with multiplicative transformations, and all inversion phases for real fields.

If I is any inversion that commutes with the Hamiltonian, and U is any multiplicative symmetry of the theory, then IU also is an inversion that transforms the « in » and « out » states in the way required by physical considerations, and which commutes with H. Furthermore, the inversions I and IU will differ only in their phase factor, according to

$$n_m^{IU} = n_m^I n_m(U) . \tag{21}$$

It is not possible to distinguish by experiment between the choice of I or IU to represent the inversion. The physical reason for this is that the matrix elements of these operators between two states differ only when the states transform differently under the multiplicative symmetry U, and transitions, either real or virtual, between such states are forbidden by the conservation of U. This fact was first pointed out by WICK, WIGHTMAN and WIGNER ([5]).

The converse of this result also holds for these inversions. That is, if a particular Hamiltonian commutes with two inversions I and I', both of which transform the free particle observables in the same way, then the Hamiltonian also commutes with the quotient operation $I^{-1}I'$, which is a multiplicative operation, and so the theory contains at least one multiplicative symmetry. This leads directly to the main problem of this Section, which is the question of what phases can arise in a physical theory, and what properties in the theory allow for the use of « unconventional » phases.

We will illustrate the discussion by referring to the parity operation, which is the most familiar and most often discussed ([6]). According to eqs. (7), (8), (9), if a theory is invariant under space reflection, it will be invariant under the multiplicative operator $\boldsymbol{P}^2 = GF$. It has sometimes been argued ([6]) that since $\boldsymbol{P}^2$ is the operator representing double reflection, it must be the identity operator for bosons, and either the identity operator or F for fermions. This is based on a principle that observable quantities should be unchanged by double reflection. It was pointed out in the fundamental paper by WICK, WIGHTMAN and WIGNER that such a principle cannot be used without some way of specifying what quantities are observable. These authors have given examples of some hermitian operators which occur in field theories and yet cannot be measured if the theory contains certain symmetries. A detailed analysis of which quantities appearing in field theories are observable would be difficult, although quite interesting. However, it appears reasonable that only such

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([5]) G. C. WICK, A. WIGHTMAN and E. P. WIGNER: *Phys. Rev.*, **88**, 101 (1952). This will be referred to as WWW.

([6]) See ref. ([5]), and also C. N. YANG and J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950); P. T. MATTHEWS: *Nuovo Cimento*, **6**, 642 (1957).

quantities that are invariant under all of the multiplicative symmetries of a theory can be observed. Since the field operators themselves are not in general invariant under multiplicative transformations, they will not be observables in theories containing such symmetries. Quantities such as the momentum, spin and charge, which are constructed from the free particle Lagrangian, are invariant under all multiplicative transformations, since they involve products like $\varphi_m^\dagger \varphi_m$. Thus, for these quanities the principle that observable quantities should be invariant under double reflection therefore does not restrict the operator $\boldsymbol{P}^2$ at all, and such restrictions can only be obtained by examining the interactions. But these will only require that the phases be chosen to give invariance of the Hamiltonian, and we will show below examples of interactions which require arbitrary phases to give invariance. We conclude that no *a priori* restrictions on the phases for space reflection, etc., can be admitted.

Suppose that the Hamiltonian commutes with a parity operator $\boldsymbol{P}$ for some choice of phases n_m^P. In general, since $\boldsymbol{P}^2 \neq 1$, the operator $\boldsymbol{P}$ will have complex eigenvalues. Thus with this choice of phases, the « intrinsic parities » of the particles created by the fields φ_m, ψ_m will be complex numbers of modulus one. We examine the circumstances under which these complex intrinsic parities can be eliminated by a redefinition of the parity operator.

Since $\boldsymbol{P}$ commutes with H, and $\boldsymbol{P}^2 \neq 1$, the theory necessarily contains at least one multiplicative symmetry $\boldsymbol{P}^2 = GF$. We consider theories invariant under rotation, which also have the multiplicative symmetry F. The general condition under which intrinsic parities may be chosen real is that $\sqrt{G^+F}$ should commute with the Hamiltonian (7). For if this happens, it is possible to define a new parity operator $\boldsymbol{P}' = \sqrt{G^+F}\,\boldsymbol{P}$, which satisfies

$$\boldsymbol{P}'^2 = G^+F\boldsymbol{P}^2 = G^\dagger I G F = 1 \tag{22}$$

and thus has real eigenvalues. Furthermore, $\boldsymbol{P}'$ commutes with H, since it is the product of operators which commute with H. Conversely, the condition is a necessary one, because if there exists a conserved parity operator $\boldsymbol{P}'$ satisfying $\boldsymbol{P}'^2 = 1$, then $\boldsymbol{P}' = U\boldsymbol{P}$, where U is a multiplicative symmetry operator, and $U^2 = G^\dagger F$. Therefore, having once found a parity operator which commutes with H, involving complex phases, it is possible to test whether the use of such phases is essential by seeing whether for the G defined by these phases, $\sqrt{G^\dagger F}$ is a symmetry of the theory. There are three general cases to be considered.

1) All of the multiplicative symmetries of the theory, including F, are parts of continuous gauge groups. This is believed to be the case in the present

(7) Here $\sqrt{G^+F}$ refers to any of the square roots defined above.

theory of elementary particles, assuming that strangeness is an additive quantum number for strong interactions, rather than a multiplicative one (8). For such theories, since $G^\dagger F'$ $(= P^{-2})$ is a number of a gauge group which commutes with the Hamiltonian, $\sqrt{G^\dagger F}$ is also a member of the gauge group and so also commutes with the Hamiltonian. It is then always possible to make the intrinsic parity of fermions and bosons real in theories satisfying assumption 1. As indicated, this is probably the case in the present theory of elementary particles.

2) The theory contains apart from gauge transformations the additional invariance F, such that $\sqrt{F}$ is not a multiplicative symmetry. In such theories there is no additive conservation of fermions, or else $\sqrt{F}$ would be part of the fermion gauge group. If there is parity conservation with $P^2 = GF'$ then since G commutes with H, there are two possibilities for G. Either

a) $G = a$ gauge transformation, γ, so that $P^2 = \gamma F'$,

or

b) $G = \gamma F'$, so that $P^2 = \gamma$.

In case *a*), the conserved operator $P' = \sqrt{\gamma^\dagger} P$ satisfies

$$P'^2 = F' .$$

Thus up to a gauge transformation, the intrinsic parities of all bosons are real, and of all fermions are imaginary in this case. An example of such a theory is given by the interaction

$$H_{\text{int}} = \overline{\psi}^C \psi \varphi + \overline{\psi} \psi \varphi + \text{h. c.} \tag{23}$$

Here φ is a real boson field and ψ a complex fermion field. It is easy to see that for invariance under P, $n_\psi = \pm 1$, $n_\varphi = 1$, so that $G = 1$, $P^2 = F'$ or the intrinsic parity of the fermion is imaginary, while that of the boson is real. Since this theory has no gauge invariances, there is no freedom in choosing these parities, except that coming from F', which accounts for the $\pm$ sign in n_ψ and makes the relative parity of the boson and fermion unobservable.

Theories containing real fermion fields and satisfying assumption 2 must fall under case *a*) if they conserve parity, since according to Section **2**, $P^2 = F'$ for such fields.

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(8) The possibility that strangeness conservation might be multiplicative was suggested by W. HEISENBERG and W. PAULI (preprint). See also K. M. CASE, R. KARPLUS and C. N. YANG: *Phys. Rev.*, **101**, 874 (1956).

In case *b*), the parity operator can again be redefined as $P' = \gamma^{\neq} P$, and $\boldsymbol{P}'^2 = 1$, so that up to a gauge transformation, all parities, fermion or boson, are real.

A Hamiltonian giving such a theory is

$$H_{\text{int}} = \overline{\psi}^{\rho} \psi \varphi + \overline{\psi} \gamma_5 \psi \varphi + \text{h. c.} \tag{24}$$

with the same symbols as before. Now $n_\psi = \pm i$, $n_\varphi = -1$, so that $G = F$, $P^2 = 1$.

It should be emphasized that if assumption 2*a* is satisfied, then all fermions have imaginary parity, whereas if 2*b* is satisfied, all have real parity, modulo gauge transformations. This type of theory does not have enough symmetry to allow some fermion to have irremovably real parity while others have irremovably imaginary parity. This is because the only non-gauge multiplicative invariance we have allowed is F, which does not distinguish between fermions.

3) The theory contains multiplicative symmetries U, other gauge transformations and F, such that $\sqrt{U^\dagger}$ and $\sqrt{U^\dagger F}$ do not commute with H. In this case, if there is parity conservation with $G = U$, then it is impossible to find a parity operator which commutes with H and satisfies $\boldsymbol{P}^2 = 1$, or $\boldsymbol{P}^2 = F$. Then we expect that the intrinsic parities of bosons and fermions might be arbitrary complex numbers, providing that the theory has sufficiently complicated multiplicative symmetries. It is clear that the existence of « discrete » multiplicative symmetries is only a necessary condition that use of complex n^P should be unavoidable in a theory, rather than sufficient. This is because there is never a conservation law for intrinsic parities alone, without specification of the orbital states involved. This is illustrated by the interactions (23) and (24) which have the same multiplicative symmetry F.

We continue the discussion by reference to a particular example. Consider the interaction of a fermion field ψ with a complex boson field φ, given by

$$H = \overline{\psi} \boldsymbol{O} \psi \varphi^2 + \overline{\psi} \overline{\boldsymbol{O}} \psi \varphi^{\dagger 2} \qquad (\overline{\boldsymbol{O}} = \gamma_4 \boldsymbol{O}^\dagger \gamma_4). \tag{25}$$

We wish to consider two cases

$$(a) \qquad \boldsymbol{O} = \gamma_5 .$$

Here the theory is invariant under space reflection transformations with the following phases

$$n_\psi^P = \text{any phase factor},$$

$$n_\varphi^P = \pm i .$$

Thus

$$\begin{cases} G\psi G^{-1} = -(n_\psi^P)^2\psi\,, \\ G\varphi G^{-1} = -\varphi\,. \end{cases} \tag{26}$$

Furthermore, the theory possesses the following multiplicative invariances:

$$B_\lambda \psi B_\lambda^{-1} = \exp[i\lambda]\psi\,, \qquad \text{for all real } \lambda \tag{27}$$

a gauge transformation on the spinor field, and

$$U\varphi U^{-1} = -\varphi$$

a discrete transformation on the boson field.

It is clear that the phase factor in the transformation of the spinor field is only conventional, and can be removed by a redefinition of the parity operator. However, the phase factor $\pm i$ for φ cannot be removed. This is because the operator $\sqrt{U^\dagger}$, defined by

$$\sqrt{U^\dagger}\varphi\sqrt{U} = \pm i\varphi\,,$$

does *not* commute with the Hamiltonian, so that $\sqrt{U^\dagger}P$ is not a conserved operator. The existence of the phase factor $\pm i$ is essential in the physical interpretation of the theory. The interaction (25) involves, among other processes, the annihilation of two S wave φ quanta together with a transition of the fermion from an S state to a P state. Such process cannot be consistent with invariance under space reflection unless the intrinsic parity of the φ quanta is $\pm i$. This follows immediately from the conservation law

$$(n_\varphi^P)^2(-1)^{\Sigma l_{\text{initial}}} = (-1)^{\Sigma l_{\text{final}}}\,. \tag{28}$$

b) Consider next $\boldsymbol{O} = 1$. The Hamiltonian has the same multiplicative invariances as before, but now it is invariant under space reflections with

$$n_\varphi^P = \text{any complex number}\,,$$
$$n_\varphi^P = \pm 1$$

and so here by a suitable redefinition of the parity operator the intrinsic parities can be made real. This indicates as stated that the existence of discrete multiplicative symmetries only allows the possibility of irremovably complex intrinsic parities, without requiring them.

Next we construct a Hamiltonian requiring $\sqrt{i}$ for the space reflection of a fermion field. To do this, consider a complex spinor field ψ interacting with a real boson field φ.

$$H_{\text{int}} = g\overline{\psi}\gamma_5\psi\varphi + h\overline{\psi}^C\psi\overline{\psi}^C\psi\varphi + \text{h. c.} \tag{29}$$

From the definition of $\overline{\psi}^C$, it is easy to see that

$$P\overline{\psi}^C P^{-1} = -in_\psi^P\gamma_4\overline{\psi}^C .$$

The interaction (29) is invariant under space reflection with

$$n_\varphi^P = -1 , \qquad n_\psi^P = \pm\sqrt{i} . \tag{30}$$

It is also invariant under the multiplicative transformations F and

$$W_\pm\psi W_\pm^{-1} = \pm i\psi .$$

However, it is not invariant under $\sqrt{W^+}$, and therefore the factors $\pm\sqrt{i}$ in the space reflection of ψ are not removable.

From these examples, it may be seen that one can construct Hamiltonians which require any n-th root of 1 as a phase factor in the transformation of complex fields under space reflection. These Hamiltonians will be characterized by the existence of discrete multiplicative invariances, whose square roots are not invariances of the theory. It is also possible to write « interactions » which require other complex phase factors, but these will involve irrational operations, on the field operators, whose meaning is questionable.

In the light of our discussion, we can conclude the following about the four classes of spinors introduced by YANG and TIOMNO ([6]), and used by many other authors. In any theory invariant under space rotations, $F = (-1)^{N_F}$ is a multiplicative symmetry. Any such theory could possibly be invariant under a parity operation in which $P^2 = F$. As we have stressed, this would mean that all fermions have parity $\pm i$. However, the invariance under F does not by itself allow the relative parity of two fermions to be imaginary. Such a possibility is connected with the existence of other discrete multiplicative symmetries, which do not act the same way on all fermion fields.

It may be further noted that the use of discrete multiplicative symmetries or of space reflection invariance to forbid unwanted processes as is sometimes done involves the difficulty that these can only give conservation laws « mo-

dulo n » and not the absolute conservation laws associated with gauge groups (9). If one accepts the usual conservation laws as absolute (conservation of charge, baryons, leptons, and strangeness in strong interactions) the use of phases $\pm i$ for some fields is unnecessary, and can be removed by a redefinition of the parity operator. On the other hand, if strangeness conservation only held modulo 4, for instance, it might be necessary to use complex phase factors for strange particles. This would happen, *e.g.*, if four Λ^0 in S states could go into three S state neutrons and one P state neutron.

We conclude this discussion with some comments about the conditions under which the relative parity of two states is measurable. Our conclusions here are in essential agreement with those of WWW. The general result may be stated as follows

The relative parity of two states is measurable only if the states transform the same way under all the multiplicative symmetry operations of the theory.

These is because, if P is a conserved parity operator, then so is UP, where U is any multiplicative symmetry operator. But if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenstates of P, with

$$\begin{cases} P|\psi_1\rangle = \varepsilon_1|\psi_1\rangle\,, \\ P|\psi_2\rangle = \varepsilon_2|\psi_2\rangle\,. \end{cases} \tag{31}$$

Then

$$\begin{cases} UP|\psi_1\rangle = \varepsilon_1 n_1(U)|\psi_1\rangle\,, \\ UP|\psi_2\rangle = \varepsilon_2 n_2(U)|\psi_2\rangle\,, \end{cases} \tag{32}$$

where

$$U|\psi_1\rangle = n_1(U)|\psi_1\rangle\,,$$
$$U|\psi_2\rangle = n_2(U)|\psi_2\rangle\,.$$

Then unless $n_1(U) = n_2(U)$ for all U, the two parity operators, which according to our previous remarks are physically indstinguishable, will have different relative eigenvalues for the two states.

Equivalently, the phase factor in the inversion of a field, or a product of fields, is measurable only if the field or product of fields is invariant under all multiplicative symmetries of the theory.

As an example of this, we note that the quantity which is measurable is the relative parity of a Ξ^-p) system compared to a 2 Λ^0 system, rather than the relative parity of Ξ and nucleon. If strangeness is an additive quantum

(9) This is true unless one assumes in addition specific forms for the interaction, such as Yukawa couplings. If the latter is done, the discrete multiplicative invariance of H may imply a continuous gauge invariance.

number, then the intrinsic parity of the Λ may be chosen real by convention, and then there is no difference in the two statements. However, if strangeness were multiplicative, the statements are not equivalent.

The above discussion of parity can also be applied to CT, CPT and all inversions of real fields, which commute with all multiplicative transformations.

5. – Products of inversions.

In this Section we discuss some of the relations among products of the inversions, and the multiplicative operators E, F, G. We consider a theory invariant under C, P and T simultaneously. Then according to Section 2, it will be invariant under E, F and G. The following results for the products of inversions can easily be demonstrated.

$$
\left\{
\begin{array}{ll}
(a) & P^2 = FG\,, \\
(b) & C^2 = 1\,, \\
(c) & T^2 = F\,, \\
(d) & (CP)^2 = (PC)^2 = F\,, \\
(e) & (TP)^2 = (PT)^2 = F\,, \\
(f) & (CT)^2 = (TC)^{-2} = FGE^{-2} \\
(g) & CP = PCG\,, \\
(h) & TP = PTFG\,, \\
(i) & CT = TCGE^{-2}\,, \\
(j) & (TCP)^2 = FE^2\,,
\end{array}
\right. \tag{33}
$$

As an example, we derive the relation $CT = TCGE^{-2}$ for a spinor field. From (1), (3)

$$
\begin{aligned}
CT\psi T^{-1}C^{-1} &= n_C n_T \gamma_5 \overline{\psi}^T\,, \\
CT\psi C^{-1}T^{-1} &= n_C^* n_T^* \gamma_5 \overline{\psi}^T\,, \\
CT\psi T^{-1}C^{-1} &= n_C^2 n_T^2 TC\psi C^{-1}T^{-1} \\
&= TCn_C^{*2} n_T^{2*} \psi C^{-1}T^{-1} \\
&= TCGE^{-2}\psi (GE^{-2})^{-1} C^{-1}T^{-1}\,, \\
CT &= TCGE^{-2}\,.
\end{aligned}
$$

Because of the relations (6) between $\boldsymbol{C}$, $\boldsymbol{T}$ and multiplicative operators, the relations (b), (c), (d), (e) cannot be changed by redefinition of $\boldsymbol{C}$, $\boldsymbol{P}$ or $\boldsymbol{T}$. Since E commutes with H, one can redefine P by

$$\boldsymbol{P}' = E^{-1}\boldsymbol{P}$$

and obtain a conserved parity operator for which

$$\boldsymbol{P}'^2 = F\mathcal{G}E^{-2} = F\mathcal{G}', \tag{33a}$$

$$(\boldsymbol{CT})^2 = F\mathcal{G}', \tag{33f}$$

$$\boldsymbol{CP}' = \boldsymbol{P}'\boldsymbol{C}\mathcal{G}', \tag{33g}$$

$$\boldsymbol{TP}' = \boldsymbol{P}'\boldsymbol{T}F\mathcal{G}', \tag{33h}$$

$$\boldsymbol{CT} = \boldsymbol{TC}\mathcal{G}', \tag{33i}$$

$$(\boldsymbol{TCP}')^2 = F. \tag{33j}$$

This is the general result in a theory containing some discrete multiplicative invariances. If, however, $\sqrt{\mathcal{G}^{+\prime}F}$ is a symmetry of the theory, then it is possible to again redefine P and C so that other relations becomes simplified,

$$P'' = \sqrt{\mathcal{G}^{+\prime}F}\,P',$$

$$C' = \sqrt{\mathcal{G}^{+\prime}F}\,C$$

and we drop primes. Then $\boldsymbol{P}$, $\boldsymbol{C}$ still commute with H and

$$\boldsymbol{P}^2 = 1, \tag{33a''}$$

$$(\boldsymbol{CT})^2 = 1, \tag{33f''}$$

$$\boldsymbol{TP} = \boldsymbol{PT}, \tag{33h''}$$

$$\boldsymbol{CP} = \boldsymbol{PC}F, \tag{33g''}$$

$$\boldsymbol{CT} = \boldsymbol{TC}F, \tag{33i}$$

$$(\boldsymbol{TCP})^2 = F. \tag{33j''}$$

It is also possible, by omitting the $\sqrt{F}$ in the definition of P'', to remove the factor F from the relations g, i at the price of restoring it to the others. The form used here is that usually adopted, while the latter is used in the Majorana neutrino theory, where $\boldsymbol{C}$ is a multiplicative operator.

* * *

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Note added in proof.

We have been informed by Dr. G. LÜDERS that problems similar to those treated in our Section 4 were discussed by him at the Summer School in Varenna in July 1959.

RIASSUNTO (*)

Si discutono i fattori di fase che possono comparire nella definizione delle inversioni $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{T}$, e i loro prodotti. Si dimostra che, a causa dell'esistenza di Hamiltoniane « fisicamente equivalenti », le fasi in ω, $\boldsymbol{CP}$, $\boldsymbol{T}$ e $\boldsymbol{TP}$ non sono misurabili per campi complessi. Per le restanti inversioni si possono costruire delle interazioni che richiedano fasi più generali delle usuali ± 1, $\pm i$ per i campi complessi; ciò è possibile se, e solo se, la teoria contiene certe discrete simmetrie moltiplicative. Si danno esempi di tali interazioni.

(*) *Traduzione a cura della Redazione.*

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La diffusion méson-nucléon dans l'état S et l'interaction méson-méson en théorie de la source fixe.

I. – Diffusion par la source.

G. Bonnevay

Ecole Normale Supérieure - Paris

(ricevuto il 22 Luglio 1959)

Summary. — On discute les valeurs des déphasages S de la diffusion méson-nucléon déduites de la théorie de la source fixe considérée comme limite de la théorie relativiste pseudoscalaire indépendante de charge, la constante de couplage étant déterminée par la diffusion P. Les amplitudes de diffusion ainsi obtenues sont trop grandes et toutes deux de même signe. Elles serviront de point de départ à un prochain travail dans lequel il sera tenu compte de la diffusion du méson incident par le nuage du nucléon cible.

1. – Introduction.

Les méthodes de Chew-Low et Wick [1] appliquées à la théorie de la source fixe, ont permis de comprendre assez clairement les principales caractéristiques de la diffusion méson-nucléon, à basse énergie, dans l'état P.

En admettant que les déphasages S sont petits, on peut retrouver leurs valeurs expérimentales à l'aide des relations de dispersion (Gilbert [2]). Mais il ne semble pas que l'on puisse expliquer leurs faibles valeurs, c'est-à-dire la faiblesse de l'interaction méson-nucléon dans l'état S, à l'aide de la seule constante de couplage f déterminée par la diffusion P.

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(1) G. F. Chew et F. E. Low: *Phys. Rev.*, **101**, 1570 (1956); G. C. Wick: *Rev. Mod. Phys.*, **27**, 339 (1955).

(2) W. Gilbert: *Phys. Rev.*, **108**, 1078 (1957).

Notre ambition serait de donner une description de la diffusion S, à basse énergie, en considérant la théorie de la source fixe comme la limite de la théorie relativiste pseudo-scalaire.

Les diagrammes de Feynman de la diffusion se classent en deux types:

a) Ceux qui ne contiennent pas d'autres lignes de nucléon que celle du nucléon source.

b) Ceux qui, en outre, comprennent des lignes de nucléon fermées ou « cycles »; de ces cycles partent $2n$ lignes de méson, ils représentent donc des interactions méson-méson. MATTHEWS et SALAM [3] ont montré que l'on devait en outre ajouter une interaction de contact entre méson, du type $\lambda[\varphi_i(x)\varphi_i(x)]^2$ afin de rendre la théorie renormalisable.

La théorie dépend alors de 4 paramètres:

1) la masse du méson μ;

2) la masse du nucléon M qui, dans l'approximation de la source fixe est en quelque sorte remplacée par la costante de cut-off m;

3) la constante de couplage $G/2M = f/\mu$; enfin

4) la constante de couplage méson-méson λ.

En fait, nous nous contenterons d'une théorie semi-phénoménologique

Nous traiterons, tout d'abord, (I-ère partie) la diffusion par la source seule diffusion représentée par l'ensemble des diagrammes de la classe *a*) ci-dessus. On retrouvera que la théorie prévoit des déphasages S trop grands et trop peu séparés; puis, dans une IIème partie (à paraître), nous étudierons les modifications apportées à la diffusion par l'interaction méson-méson en schématisant celle-ci par une simple interaction de contact $\lambda[\varphi_i(x)\varphi_i(x)]^2$, et nous verrons s'il est possible de choisir λ de façon à rétablir l'accord avec l'expérience.

Pour étudier la diffusion par la source seule, nous utiliserons les équations de Drell, Friedman et Zachariasen [4] qui sont une généralisation de l'équation de Chew et Low au cas d'interactions non-linéaires par rapport au champ mésique.

Le hamiltonien d'interaction du méson avec la source fixe est donné, par exemple, par celui de FOLDY [5], mais DRELL et ses collaborateurs n'en con-

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[3] P. T. MATTHEWS: *Phys. Rev.*, **80**, 292 (1950); *Phil. Mag.*, **42**, 211 (1951); *Phys. Rev.*, **81**, 936 (1951); P. T. MATTHEWS et A. SALAM: *Phys. Rev.*, **94**, 185 (1954).

[4] S. DRELL, M. FRIEDMAN et F. ZACHARIASEN: *Phys. Rev.*, **104**, 236 (1956).

[5] L. FOLDY: *Phys. Rev.*, **84**, 168 (1951).

servent que les trois premiers termes:

$$(1.1)\qquad \begin{cases} H_1 = H_P + H_S^1 + H_S^2 , \\ H_P = \dfrac{f^0}{\mu}\int \boldsymbol{\sigma}\nabla\boldsymbol{\tau}_i\varphi_i(x)s(x)\,\mathrm{d}^3x , \\ H_S^1 = g_0^0\int \varphi_i(x)\varphi_i(x')s(x)s(x')\,\mathrm{d}^3x\,\mathrm{d}^3x' , \\ H_S^2 = g^0\int \tau_i[\boldsymbol{\varphi}(x)\wedge\boldsymbol{\pi}(x')]_i s(x)\,s(x')\,\mathrm{d}^3x\,\mathrm{d}^3x' , \end{cases}$$

où $\varphi_i(x)$ est l'opérateur champ mésique (i l'indice de spin isotopique), f^0 étant la constante d'interaction méson-nucléon, et [6]

$$(1.2)\qquad g_0^0 = 2M^0\left(\frac{f^0}{\mu}\right)^2 , \qquad g^0 = \left(\frac{f^0}{\mu}\right)^2 ,$$

Les équations de Drell, Friedman et Zachariasen donnent les amplitudes de diffusion dans l'état P et l'état S en fonction, non pas des constantes f^0, g_0^0 et g^0, mais de leurs valeurs renormalisées

$$(1.3)\qquad \begin{cases} f = \dfrac{\langle\psi_0|\sigma_\alpha\tau_i|\psi_0\rangle}{\langle u_0|\sigma_\alpha\tau_i|u_0\rangle} f^0 = \varrho_1 f^0 , \\ g_0 = \dfrac{\langle\psi_0|\psi_0\rangle}{\langle u_0|u_0\rangle} g_0^0 = g_0^0 , \\ g = \dfrac{\langle\psi_0|\tau_i|\psi_0\rangle}{\langle u_0|\tau_i|u_0\rangle} g^0 = \varrho_2 g^0 , \end{cases}$$

où $|u_0\rangle$ est le vecteur d'état du nucléon « nu » (vecteur propre du hamiltonien libre), $|\psi_0\rangle$ le nucléon physique, c'est-à-dire en présence de son nuage mésique (vecteur propre du hamiltonien total d'énergie la plus basse).

En faisant l'approximation à un méson, DRELL et ses collaborateurs ont résolu leurs équations en prenant g_0 et g comme des paramètres arbitraires qu'ils ont ajustés de façon à retrouver les déphasages S expérimentaux donnés par Orear [7]:

$$(1.4)\qquad \delta_1 = 0.16\,\frac{\hbar k}{\mu C} , \qquad \delta_3 = -\,0.11\,\frac{\hbar k}{\mu C} .$$

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[6] L'indice zéro placé en haut d'une lettre signifie une grandeur *non-renormalisée*.
[7] J. OREAR: *Phys. Rev.*, **100**, 288 (1955).

Ils trouvent alors:

$$g_0 = 0.4\,\mu^{-1}\,, \qquad g = 0.4\,\mu^{-2}\,, \tag{1.5}$$

f étant déterminé par la diffusion P.

Notre but est différent: nous voudrions pouvoir rendre compte des expressions de diffusion à l'aide des seules constantes de couplage f et λ en tenant compte de (1.2, 3) pour déterminer g_0 et g.

La Section **2** du présent travail sera donc consacrée à la détermination des constantes renormalisées g_0 et g en fonction de f, ce qui peut se faire seulement grâce à une étude de l'état fondamental méson-nucléon.

A la Section **3** suivante, nous chercherons à résoudre les équations de Drell, Friedman et Zachariasen pour les valeurs g_0 et g précédemment déterminées. Nous aurons ainsi les amplitudes de diffusion par la source seule.

Nous savons que, à la limite non relativiste, l'interaction pseudo-scalaire n'est pas équivalente aux trois hamiltoniens donnés en (1.1) avec les relations (1.3), mais à une série de termes en puissance de G^0 dont ces hamiltoniens ne constituent que les premiers termes. Cependant Horwitz [8] a montré qu'en se restreignant à l'approximation à un méson dans les états intermédiaires, le hamiltonien exact conduisait aux mêmes équations que celles de Drell, Friedman et Zachariasen; seules, sont modifiées les relations (1.2) définissant les constantes renormalisées. Nous admettrons que nous obtenons encore le bon ordre de grandeur en gardant ces relations.

2. – Détérmination des constantes de couplage renormalisées.

Tout d'abord, remarquons que $g_0^0 = g_0$, en raison de la normalisation de l'état fondamental méson-nucléon avec ou sans interaction.

Pour calculer les éléments de matrice $\langle \psi_0 | \sigma_\alpha \tau_i | \psi_0 \rangle$ et $\langle \psi_0 | \tau_i | \psi_0 \rangle$, c'est-à-dire pour déterminer l'état fondamental méson-nucléon $|\psi_0\rangle$, nous avons utilisé l'approximation du « couplage intermédiaire » due à Tomonaga [9].

Christian, Friedman et Lee [10] ont déjà calculé ces éléments de matrice à l'aide de l'approximation de Tomonaga en supposant tous les mésons du nuage dans l'état P, c'est-à-dire en ne gardant que le hamiltonien H_p. On trouve alors:

$$\begin{cases} \varrho_1 = \dfrac{\langle \psi_0 | \sigma_\lambda \tau_i | \psi_0 \rangle}{\langle u_0 | \sigma_\lambda \tau_i | u_0 \rangle} = 0.381\,, \\ \varrho_2 = \dfrac{\langle \psi_0 | \tau_i | \psi_0 \rangle}{\langle u_0 | \tau_i | u_0 \rangle} = 0.073\,, \end{cases} \tag{2.1}$$

[8] L. Horwitz: *Phys. Rev.*, **108**, 886 (1957).
[9] S. Tomonaga: *Prog. Theor. Phys.*, **2**, 6 (1947).
[10] R. Christian, M. Friedman et T. D. Lee: *Phys. Rev.*, **100**, 1494 (1955).

pour une valeur de $f = 0.105$ ajustée pour rendre compte de la diffusion méson-nucléon dans l'état P. Par ailleurs, la self-énergie du nucleon étant égale à ([11]) $E = -20.6\,\mu$, on en déduit la masse non renormalisée $M^0 = M - E = 27{,}4\,\mu$; d'où les valeurs des constantes renormalisées

$$g_0 = 39\,\mu^{-1}\,, \qquad g = 0.052\,\mu^{-2}\,. \tag{2.2}$$

Comment les hamiltoniens $H_s^{1,2}$ modifient-ils ces résultats ([12])? Il est facile de voir qu'à l'approximation de Tomonaga les équations pour l'état S et pour l'état P se séparent. Les rapport ϱ_1 et ϱ_2 ne sont pas modifiés, il en est donc de même de g.

Par contre, la self-énergie due aux mésons dans l'état S est positive et diminue considérablement la trop grande valeur de M^0 qui rend g_0 trop grand. g_0^0, et M^0 sont alors solutions d'un système d'équations transcendantes que nous avons résolues graphiquement d'une manière très approchée (en particulier en négligeant g^0 dans le calcul de la self-énergie ([13])). Comme nous le verrons, les déphasages S dépendent peu de la valeur exacte de g_0^0 lorsque cette constante est grande.

On trouve finalement, en prenant $f = 0.10$,

$$g_0 = 9.68\,\mu^{-1}\,, \qquad g = 0.052\,\mu^{-2}\,. \tag{2.3}$$

Un aussi grand désaccord avec les valeurs trouvées par DRELL et ses collaborateurs, valeurs que l'on peut considérer comme imposées par les résultats expérimentaux, ne peut provenir d'un défaut de l'approximation de Tomonaga. Les règles de sommes de CINI et FUBINI ([14]) peuvent nous convaincre de l'excellence de cette approximation dans le cas présent. Ces règles établies pour une interaction H_p, sans aucune approximation, relient les rapports ϱ_1 et ϱ_2 donnés par (2.1) et la constante de couplage f à des intégrales sur les sections efficaces $\sigma(\pi^+\mathrm{p})$, $\sigma(\pi^-\mathrm{p})$ que l'on peut tirer de l'expérience. Les valeurs données en (2.1) sont alors tout à fait compatibles avec celles que fournissent les règles de somme.

2·2. *L'état fondamental méson-nucléon et l'interaction méson-méson.* – Nous avons voulu nous rendre compte de l'effet de l'interaction méson-méson sur les rapports ϱ_1 et ϱ_2. Nous avons utilisé une méthode très grossière consistant à

([11]) Valeur de E citée par R. STROFFOLINI: *Phys. Rev.*, **104**, 1146 (1956).

([12]) Travail fait en collaboration avec J. MANDELBROJT.

([13]) On est alors ramené au modéle de G. WENTZEL: *Phys. Rev.*, **93**, 233 (1954); W. THIRRING: *Helv. Phys. Acta*, **28**, 344 (1955); E. ARNOUS: *Journ. Phys. et Rad.*, **17**, 107 (1956); A. CHEVALIER et G. RIDEAU: *Nuovo Cimento*, **10**, 228 (1958).

([14]) M. CINI et S. FUBINI: *Nuovo Cimento*, **11**, 142 (1954); **3**, 764 (1956).

faire l'approximation de Tomonaga, l'interaction étant

$$H_P + \lambda \int [\varphi_i(x)\,\varphi_i(x)]^2 \, \mathrm{d}^3x \,,$$

c'est-à-dire à supposer tous les mésons dans un même état moyen (état P), puis à ne garder que les états à deux mésons intermédiaires (approximation de Tamm-Dancoff à deux mésons).

En supposant tout d'abord $\lambda = 0$, on peut comparer les résultats ainsi obtenus avec ceux de Christian, Friedman et Lee [10].

Le Tableau I nous montre que l'approximation de Tamm-Dancoff à deux mésons (notée T.D.) donne le bon ordre de grandeur pour ϱ_1, ϱ_2 et la self-énergie E due aux mésons P (les résultats de Christian, Friedman et Lee sont inscrits dans les colonnes C.F.L.).

Tableau I.

f	0	0.105		∞		
		T.D.	C.F.L.	T.D.	C.F.L.	Théorie exacte
ϱ_1	1	0.456	0.381	0.363	+ 0.323	–
ϱ_2	1	0.184	0.075	0.052	— 0.014	0
E	0	$-17.8\,\mu$	$-20.6\,\mu$	—	—	—

On peut donc espérer que la méthode donne encore le bon ordre de grandeur pour $\lambda \neq 0$.

Le résultat est que pour $\lambda > 0$, ϱ_1 et ϱ_2 *sont remarquablement indépendants de* λ. Pour $\lambda < 0$, ils tendent vers 1 lorsque $\lambda \to -\infty$.

Nous adopterons donc finalement pour g_0 et g les valeurs (2.3).

3. – Résolution des équations de Drell, Friedman et Zachariasen.

3·1. *Les équations.* – Ecrivons le hamiltonien total:

$$H = H_0 + H_1 \,,$$

$$H_0 = \sum_k a_{ik}^+ a_{ik} \,,$$

où a_{ik}^+ est l'opérateur de création d'un méson dans l'état d'impulsion $\boldsymbol{k}$ et de spin isotopique i.

A la suite de Chew, Low et Wick, Drell et ses collaborateurs [4] définis-

sent l'amplitude de diffusion par:

$$T^{ij}_{pq} = \langle \psi^{(-)}_{ip} | V | \psi_0 \rangle , \tag{3.1}$$

où

$$V_q = [H_1, a^+_{iq}]; \tag{3.2}$$

$|\psi_0\rangle$ et $|\psi^{(-)}_{ip}\rangle$ sont des vecteurs propres de H correspondant à l'état fondamental du système méson-nucléon, pour le premier, et à un état de diffusion correspondant à une onde *entrante*, pour le second.

Dans le cas d'une interaction non linéaire en $\varphi(x)$, il est nécessaire d'introduire une nouvelle amplitude:

$$S^{ij}_{pq} = -\langle \psi^{(-)}_{ip} | V^+_{jq} | \psi_0 \rangle , \tag{3.3}$$

T_{pq} et S_{pq} sont alors donnés par

$$\begin{cases} T_{pq} = \langle \psi_0 | [a_p, V_q] | \psi_0 \rangle - \sum\limits_n \left\{ \dfrac{T^*_{np} T_{nq}}{E_n - \omega_q - i\varepsilon} + \dfrac{S^*_{nq} S_{np}}{E_n + \omega_p} \right\} , \\ S_{pq} = \langle \psi_0 | [a_p, V^+_q] | \psi_0 \rangle - \sum\limits_n \left\{ \dfrac{T^*_{np} S_{nq}}{E_n - \omega_p - i\varepsilon} + \dfrac{T^*_{nq} S_{np}}{E_n + \omega_p} \right. , \end{cases} \tag{3.4}$$

qui sont la généralisation de l'équation de Low au cas des interactions non linéaires.

T_{np} et S_{nq} sont des généralisations de (3.1, 3) où l'on remplace l'état $|\psi^{(-)}_p\rangle$ par un état à n mésons $|\psi^{(-)}_n\rangle$ dont l'énergie est E_n.

En se bornant à l'approximation à 1 méson, c'est-à-dire au cas où les T_{np} sont considérés comme n'apportant aucune contribution à T_{pq} à moins que $n = 0$ ou 1, les égalités (3.4) deviennent de véritables équations pour T_{pq} et S_{pq}, les seconds membres ne contenant que T_{pq}, S_{pq} et des fonctions de $\boldsymbol{p}$ et $\boldsymbol{q}$ soit: T_{0p}, S_{0p}, $\langle \psi_0 | [a_p V_q] | \psi_0 \rangle$ qui sont directement reliées aux constantes renormalisées (1.2).

De plus, les équations se séparent en équations pour les états P et S dans le cas du hamiltonien d'interaction donné par (1.1).

Les équations pour l'état P se réduisent à une seule équation qui est l'équation de Chew et Low.

Pour l'état S, comme pour l'état P, les équations se séparent en états de spin isotopique en posant:

$$\begin{cases} T^{ij}_{pq} = \dfrac{2\pi}{\sqrt{\omega_p \omega_q}} v_p v_q Q^{ij}_\alpha t^\alpha(\omega_p, \omega_q) , \\ S^{ij}_{pq} = \dfrac{2\pi}{\sqrt{\omega_p \omega_q}} v_p v_q Q^{ij}_\alpha s^\alpha(\omega_p, \omega_q) , \end{cases} \tag{3.5}$$

les Q_α^{ij} étant les projections sur les états de spin isotopique total α du système méson-nucléon: $\alpha = 1, 3$ correspondant aux spin isotopiques $\frac{1}{2}$ et $\frac{3}{2}$.

DRELL et ses collaborateurs définissent alors deux combinaisons linéaires de $t^\alpha(\omega_p, \omega_q)$, $s^\alpha(\omega_p, \omega_q)$ qui, grâce à la forme des hamiltoniens $H_s^{1,2}$, ne sont fonctions que des énergies finales ω_p:

$$(3.6)\qquad \begin{cases} 2t^\alpha(\omega_p, \omega_q) = \dfrac{\omega_q}{\omega_p}[a_\alpha(\omega_p) + b_\alpha(\omega_p)] + a_\alpha(\omega_p) - b_\alpha(\omega_p)\,, \\ 2s^\alpha(\omega_p, \omega_q) = \dfrac{\omega_q}{\omega_p}[a_\alpha(\omega_p) + b_\alpha(\omega_p)] - a_\alpha(\omega_p) + b_\alpha(\omega_p)\,. \end{cases}$$

Les amplitudes $t^\alpha(\omega_p, \omega_q)$ et $a_\alpha(\omega_p)$ deviennent identiques sur la couche d'énergie; $b_\alpha(\omega_p)$ n'a pas de signification physique simple.

Les définitions (3.5) et (3.6) permettent de remplacer le système (3.4) par le système d'équations auxquelles satisfont les amplitudes $a_\alpha(\omega_p)$ et $b_\alpha(\omega_p)$:

$$(A)\qquad \begin{cases} (a)\quad a_\alpha(\omega) = C_0 + C_1\omega\Gamma_\alpha - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty v_k^2\,\frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\left\{\frac{|a_\alpha(\omega_k)|^2}{\omega_k - \omega - i\varepsilon} + A_{\alpha\beta}\,\frac{|a_\beta(\omega_k)|^2}{\omega_k + \omega}\right\}, \\ (b)\quad b_\alpha(\omega) = -C_0 + iD_0\omega + D_1\omega\Gamma_\alpha - \\ \qquad\qquad - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty v_k^2\,\frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\left\{\frac{a_\alpha^*(\omega_k)b_\alpha(\omega_k)}{\omega_k - \omega - i\varepsilon} + A_{\alpha\beta}\,\frac{a_\beta(\omega_k)b_\beta^*(\omega_k)}{\omega_k + \omega}\right\}, \end{cases}$$

les constantes C_0, C_1, D_0, D_1 étant liées aux constantes de couplage renormalisées (1.2) par:

$$(A)\qquad \begin{cases} (c_0)\quad C_0 = \dfrac{g_0}{2\pi} - \dfrac{1}{4\pi}\dfrac{2}{3}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k}\,v_k^2\widetilde{V}_\beta\,|a_\beta(\omega_k) - b_\beta(\omega_k)|^2\,, \\ (c_1)\quad C_1 = \dfrac{g}{\pi} + \dfrac{1}{4\pi}\dfrac{4}{3}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\,r_k^2\widetilde{W}_\beta\{|a_\beta(\omega_k) - b_\beta(\omega_k)|^2 + 2\,|a_\beta(\omega_k)|^2 - 2\,|b_\beta(\omega_k)|^2\}\,, \\ (d_0)\quad D_0 = -\dfrac{1}{\pi}\dfrac{2}{3}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\,v_k^2\widetilde{V}_\beta\,\mathrm{Im}\,[a_\beta^*(\omega_k)b_\beta(\omega_k)]\,, \\ (d_1)\quad D_1 = -\dfrac{1}{4\pi}\dfrac{4}{3}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\,v_k^2\widetilde{W}_\beta\,|a_\beta(\omega_k) - b_\beta(\omega_k)|^2\,, \end{cases}$$

où v_k est la transformée de Fourier de la fonction source

$$v_k = \frac{1}{(2\pi)^3}\int \exp[i\boldsymbol{k}\boldsymbol{x}]s(x)\,\mathrm{d}^3x\,;$$

la masse du méson, μ, est prise pour unité; $A_{\alpha\beta}$ est une matrice carrée, Γ_α une matrice-colonne, $\tilde{V}_\beta$ et $\tilde{W}_\beta$ des matrices-ligne

$$A_{\alpha\beta} = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}, \qquad \Gamma_\alpha = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}, \qquad \begin{aligned} \tilde{V}_\beta &= (1 \quad 2)\,, \\ \tilde{W}_\beta &= (1 \quad -1)\,. \end{aligned}$$

Nous devons résoudre ces équations dans des conditions très différentes de celles rencontrées par DRELL, FRIEDMAN et ZACHARIASEN. Nous avons ici une grande valeur pour g_0 et une très faible valeur pour g: on est donc conduit à poser:

$$\text{(3.7)} \qquad \begin{cases} a_\alpha(\omega) = a(\omega) + \dfrac{g}{\pi}\, a'_\alpha(\omega)\,, \\ b_\alpha(\omega) = b(\omega) + \dfrac{g}{\pi}\, b'_\alpha(\omega)\,, \\ C_0 \quad = c_0 + \dfrac{g}{\pi}\, c'_0\,, \qquad D_0 = d_0 + \dfrac{g}{\pi}\, d'_0\,, \\ C_1 \quad = \dfrac{g}{\pi}\, c'_1\,, \qquad D_1 = \dfrac{g}{\pi}\, d'_1\,. \end{cases}$$

Le système (A) se décompose alors en deux systèmes:

$$\text{(B)} \qquad \begin{cases} (a) \quad a(\omega) = c_0 - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty \frac{k}{\omega_k^2}\, v_k^2\, |a(\omega_k)|^2 \frac{2\omega_k\,\mathrm{d}\omega_k}{\omega_k^2 - \omega^2 - i\varepsilon}\,, \\ (b) \quad b(\omega) = -c_0 + i\, d_0\omega - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \left\{\frac{a^*(\omega_k)b(\omega_k)}{\omega_k - \omega - i\varepsilon} + \frac{a(\omega_k)b^*(\omega_k)}{\omega_k + \omega}\right\}, \\ (c) \quad c_0 \quad = \dfrac{g_0}{2\pi} - \dfrac{1}{2\pi}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2\, |a(\omega_k) - b(\omega_k)|^2\,, \\ (d) \quad d_0 \quad = -\dfrac{2}{\pi}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, \mathrm{Im}\,[a^*(\omega_k)b(\omega_k)]v_k^2\,, \end{cases}$$

et d'autre part:

$$
\text{(C)}\quad
\begin{cases}
(a)\quad a'_\alpha(\omega) = c'_0 + c'_1\omega\Gamma_\alpha - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \Big\{ \frac{2\,\mathrm{Re}\,[a^*(\omega_k)a'_\alpha(\omega_k)]}{\omega_k - \omega - i\varepsilon} + \\
\qquad + A_{\alpha\beta}\,\frac{2\,\mathrm{Re}\,[a^*(\omega_k)a'_\alpha(\omega_k)]}{\omega_k + \omega}\Big\} - \dfrac{\omega^2}{\pi}\dfrac{g}{\pi}\displaystyle\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \Big\{ \frac{|a'_\alpha(\omega_k)|^2}{\omega_k - \omega - i\varepsilon} + A_{\alpha\beta}\,\frac{|a'_\beta(\omega_k)|^2}{\omega_k + \omega}\Big\}, \\
(b)\quad b'_\alpha(\omega) = -c'_0 + id'_0\omega + d'_1\omega\Gamma_\alpha - \dfrac{\omega^2}{\pi}\displaystyle\int_1^\infty \frac{k^2\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \cdot \\
\qquad \cdot\Big\{ \frac{a^*(\omega_k)b'_\alpha(\omega_k) + a'^*_\alpha(\omega_k)b(\omega_k)}{\omega_k - \omega - i\varepsilon} + A_{\alpha\beta}\,\frac{a(\omega_k)b'^*_\beta(\omega_k) + a'_\beta(\omega_k)b^*(\omega_k)}{\omega_k + \omega}\Big\} + \\
\qquad - \dfrac{\omega^2}{\pi}\dfrac{g}{\pi}\displaystyle\int_1^\infty \frac{k^2\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \Big\{ \frac{a'^*_\alpha(\omega_k)b'_\alpha(\omega_k)}{\omega_k - \omega - i\varepsilon} + A_{\alpha\beta}\,\frac{a'_\beta(\omega_k)b'_\beta(\omega_k)}{\omega_k + \omega}\Big\}, \\
(c)\quad c'_0 = -\dfrac{1}{2\pi}\dfrac{1}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2 \widetilde{V}_\beta\, 2\,\mathrm{Re}\,[(a-b)^*(a'_\beta - b'_\beta)] + \\
\qquad - \dfrac{1}{2\pi}\dfrac{1}{3}\dfrac{g}{\pi}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2 V_\beta\, |a'_\beta - b'_\beta|^2], \\
(c_1)\quad c'_1 = 1 + \dfrac{1}{\pi}\dfrac{1}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \widetilde{W}_\beta\, 2\,\mathrm{Re}\,\{(a-b)(a'_\beta - b'_\beta)^* + 2a^*a'_\beta - 2b^*b'_\beta\} + \\
\qquad + \dfrac{1}{\pi}\dfrac{1}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \widetilde{W}_\beta\{|a'_\beta - b'_\beta|^2 + 2\,|a'_\beta|^2 - 2\,|b'_\beta|^2\}, \\
(d)\quad d'_0 = -\dfrac{1}{\pi}\dfrac{2}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \widetilde{V}_\beta\,\mathrm{Im}\,\{a^*b'_\beta + a'^*_\beta b\} - \dfrac{1}{\pi}\dfrac{g}{\pi}\dfrac{2}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2 \widetilde{V}_\beta\,\mathrm{Im}\,(a'^*_\beta b'_\beta), \\
(d_1)\quad d'_1 = -\dfrac{1}{\pi}\dfrac{1}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \widetilde{W}_\beta\, 2\,\mathrm{Re}\{(a-b)^*(a'_\beta - b'_\beta)\} - \\
\qquad - \dfrac{1}{\pi}\dfrac{g}{\pi}\dfrac{1}{3}\displaystyle\int \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2 \widetilde{W}_\beta\,|(a'_\beta - b'_\beta)|^2 .
\end{cases}
$$

Les deux systèmes (B) et (C) sont équivalents à (A). Si on fait $g = 0$ dans (C), on obtient par (3.7) la Ière approximation dans le developpement de a_α et b_α en puissances de g/π. Notons que ce développement est très différent de celui des perturbations puisque g est la *constante renormalisée.*

3·2. *Solution du système* (B). – Nous n'entendons pas discuter ce système et trouver toutes ses solutions. Nous cherchons la solution qui tend vers zéro avec g_0.

Remarquons tout d'abord que nous connaissons la solution exacte de ce

problème puisque, dans le cas présent, nous avons affaire au modèle de Wentzel ([13]). Voyons ce que donne l'équation de Drell, Friedman et Zachariasen.

Tout d'abord remarquons que dans le cas de la seule interaction H_S^1, V_p est antihermitique et qu'on a donc, d'après (3.1, 2, 3):

$$S_{pq}^{ij} = - T_{pq}^{ij} . \tag{3.8}$$

Il s'ensuit, d'après les définitions (3.5, 6), que

$$b(\omega) = - a(\omega) . \tag{3.9}$$

On voit immédiatement que l'équation intégrale (B-a) ètant supposée résolue, le système a pour solution $b(\omega) = - a(\omega)$ et $d_0 = 0$. On peut montrer que, pour un $a(\omega)$ donné, cette solution est unique ([15]).

Enfin, la comparaison de (B.a) et (B.c) compte tenu de (3.9), conduit à:

$$\frac{g_0}{2\pi} = a(\infty) . \tag{3.10}$$

L'équation (B.a) se ramène à une équation du type de Low soluble en posant:

$$x = \omega^2 , \qquad \psi(x) = \frac{a(\omega)}{\omega^2} \tag{3.11}$$

$$\psi(x) = \frac{c_0}{x} - \frac{1}{\pi} \int_1^\infty k_y v_y^2 \frac{\psi(y)^2}{y - x - i\varepsilon} \, y \, dy . \qquad k_y = \sqrt{y - 1} : \tag{3.12}$$

Supposons que nous ayons trouvé une solution de (3.12), que nous appellerons $\psi(x)$; nous pouvons la prolonger dans le plan complexe par:

$$\psi(z) = \frac{c_0}{z} - \frac{1}{\pi} \int_1^\infty k_y v_y^2 \frac{|\psi(y_+)|^2}{y - z} \, y \, dy . \tag{3.13}$$

Une méthode aujourd'hui classique ([16]) permet, en étudiant un ensemble de propriétés nécessaires et suffisantes que doit remplir la fonction analytique

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([15]) En utilisant la forme (3.24) pour $a(\omega)$, on ramène les équations (B.b), (3.27) au type des équations de Mushkelishvili. Il est alors facile de voir que les équations homogènes correspondantes n'ont pas de solution. (Voir N. MUSHKELISHVILI: *Trud. Tbilissi. Mat. Inst.*, **10**, 1 (1941); S. G. MIKHLIN: *Integral Equations* (London, 1957); ces équations ont été utilisées en *Théorie des Champs* par R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1958); voir également M. GOURDIN et A. MARTIN: *Nuovo Cimento*, **8**, 699 (1958).

([16]) J. CASTILLEJO, G. DALITZ et F. DYSON: *Phys. Rev.*, **101**, 453 (1956).

$\psi(z)$, de montrer que cette fonction, si elle existe, est nécessairement de la forme (pour la solution tendant vers la série des perturbations)

$$\psi(z) = \frac{1}{z}\frac{1}{h(z)}, \tag{3.14}$$

avec:

$$h(z) = \alpha + \frac{1}{\pi}\int_1^\infty kv^2 \frac{dy}{y-z}, \quad \alpha = \frac{1}{c_0} - \frac{1}{\pi}\int_1^\infty kv^2 \frac{dy}{y} . \tag{3.15}$$

La solution de (3.12) est alors:

$$\psi(x) = \psi(x_+) = \lim_{\varepsilon \to 0} \psi(x + i\varepsilon) . \tag{3.16}$$

Pour que (3.14, 15, 16) soit effectivement solution, il est alors nécessaire que $\psi(z)$ n'ait pas d'autre pôle que le pôle simple $1/z$ (voir (3.13)), c'est-à-dire que $h(z)$ n'ait pas de zéro.

Remarquons que

$$\operatorname{Im} h(z) = \operatorname{Im} z \frac{1}{\pi}\int_1^\infty k_y v_y^2 \frac{dy}{|y-z|^2},$$

ne peut s'annuler que si $\operatorname{Im} z = 0$. Les zéros de $h(z)$ sont donc sur la partie de l'axe réel où $h(z)$ existe c'est-à-dire en dehors de la ligne de singularité $1 < x < \infty$ (si v_k ne s'annule qu'à l'infini).

Or, pour ces valeurs de x

$$\frac{dh(x)}{dx} = \frac{1}{\pi}\int_1^\infty k_y v_y^2 \frac{dy}{|y-x|^2} > 0 , \qquad \text{si } x<1, \tag{3.17}$$

mais

$$h(1) = \frac{1}{c_0} + \frac{1}{\pi}\int_1^\infty k_y v_y^2 \frac{dy}{y(y-1)} > 0 . \tag{3.18}$$

Les inégalités (3.16, 17) entraînent la condition nécessaire et suffisante pour que $h(z)$ n'ait pas de zéro:

$$h(\infty) = \alpha > 0$$

d'où

$$c_0 < \frac{1}{(1/\pi)\int_1^\infty kv^2 dy/y}, \tag{3.19}$$

d'après (3.11, 14, 15, 16), la solution de (B.a) s'annulant avec g_0 s'écrit donc:

$$a(\omega) = \frac{c_0}{1 + c_0[f(\omega_+^2) - f(0)]} \tag{3.20}$$

avec

$$f(\omega_+^2) = \frac{1}{\pi}\int_1^\infty k_y v_y^2 \frac{\mathrm{d}y}{y - \omega^2 - i\varepsilon},$$

et la condition (3.19) qu'on peut aussi écrire:

$$c_0 < \frac{1}{f(0)}. \tag{3.21}$$

La relation (3.10) permet de plus d'exprimer $c_0 = a(0)$ et $a(\omega)$ en fonction de la constante de couplage

$$g_0' = \frac{g_0}{2\pi} = a(\infty), \tag{3.22}$$

$$a(\omega) = \frac{g_0'}{1 + g_0' f(\omega_+^2)}, \qquad b(\omega) = -a(\omega), \; c_0 = a(0). \tag{3.23}$$

Ceci n'est pas autre chose que la solution exacte du modèle de Wentzel. On voit donc que, dans ce cas, les équations du type de Low à l'approximation à un méson conduisent à la solution exacte.

Les déphasages S. – Négliger le couplage g revient à négliger les termes en μ/M et, comme le montre le theorème de croisement, on retrouve bien que les déphasages de l'état S deviennent égaux: $\delta_1 = \delta_3$.

D'après l'égalité:

$$\operatorname{Im} a(\omega_p) = -pv_p^2 |a(\omega_p)|^2,$$

que l'on peut tirer de (B.a), et qui traduit l'unitarité de la matrice S, il existe une fonction $\delta(p)$ (le déphasage) telle que:

$$a(\omega) = -\frac{\exp[i\delta(p)] \sin\delta(p)}{pv_p^2}, \tag{3.24}$$

et, d'après (3.20):

$$\operatorname{tg}\delta(p) = -\frac{pv_k^2}{h(p)}, \tag{3.25}$$

où $\bar{h}(p)$ est la partie réelle de $h(p)$ (3.15)

$$\bar{h}(p) = \frac{1}{c_0} + \frac{1}{\pi} P \int_1^\infty k v_k^2 \frac{d\omega_k^2}{\omega_k^2 - \omega_p^2} - \frac{1}{\pi} \int_1^\infty k v_p^2 \frac{d\omega_k^2}{\omega_k^2} .$$

On voit donc que:

$$\operatorname{tg} \delta(0) = \operatorname{tg} \delta(\infty) = 0$$

et que tg δ ne peut changer de signe entre 0 et l'∞. Nous choisirons la détermination de $\delta(p)$ telle que:

$$\delta(0) = \delta(\infty) = 0 .$$

Remarque. – Pour une fonction de cut-off « carrée », c'est-à-dire:

$$v_k = \begin{cases} 1 & \text{pour } 0 < k < K \\ 0 & \text{pour } K < k \end{cases}$$

l'équation (B.a) n'a plus de solution quel que soit K car la condition (3.19) doit alors être remplacée par

$$c_0 < \frac{-1}{\bar{h}(K)} ,$$

mais $\bar{h}(K) = -\infty$ pour tout K.

De la même façon, si $v_k = 1$ (source ponctuelle) le problème n'a plus de solution.

Cependant, dans ces deux cas, la fonction $a(\omega)$ (3.20) continue à exister.

D'une manière générale, lorsque la condition (3.19) n'est plus remplie, (3.20) définit une amplitude constituant toujours une solution formelle du problème de Wentzel, mais l'équation (B.a) n'étant plus vérifiée, la condition d'unitarité n'est plus remplie et l'on voit apparaître des comportements anormaux de l'amplitude, sans interprétation physique (« fantômes ») (cf. par ex. Arnous, Rideau [13]).

3·3. *Solution du systeme* (C). – Les équations intégrales donnant les 4 fonctions $a'_\alpha(\omega)$ et $b'_\alpha(\omega)$ se résolvent exactement si $g = 0$ (premier ordre de l'approximation des perturbations en g pour le système (A)).

3·3.1. Résolution des équations en $a'_\alpha(\omega)$. – Les deux premières équations de (C) se séparent, si $g = 0$, en posant

(3.26) $$a'_\alpha(\omega) = a^{(0)}(\omega) + \Gamma_\alpha a^{(1)}(\omega)\omega ,$$

les deux fonctions $a^{(0)}(\omega)$ et $a^{(1)}(\omega)$ sont alors solutions de l'équation:

$$a^i(\omega) = c'_i - \frac{\omega^2}{\pi}\int_1^\infty \frac{k v_k^2}{\omega_k^2}\, 2\,\mathrm{Re}\,[a^*(\omega)a^i(\omega)]\,\frac{2\omega_k \mathrm{d}\omega_k}{\omega_k^2 - \omega^2 - i\varepsilon}, \tag{3.27}$$

avec $i = 0$ ou 1.

On obtient une solution de (3.27) en considérant, dans l'équation (B.a), la constante c_0 comme fonction d'un paramètre λ et en dérivant les deux membres de (B.a) par rapport à λ

$$\frac{\mathrm{d}a_\lambda(\omega)}{\mathrm{d}\lambda} = \frac{\mathrm{d}c_0(\lambda)}{\mathrm{d}\lambda} - \frac{\omega^2}{\pi}\int_1^\infty \frac{k v_k^2}{\omega_k^2}\, 2\,\mathrm{Re}\left[a_\lambda^*(\omega)\,\frac{\mathrm{d}a_\lambda(\omega_k)}{\mathrm{d}\lambda}\right]\frac{2\omega_k \mathrm{d}\omega_k}{\omega_k^2 - \omega^2 - i\varepsilon},$$

qui montre que $\mathrm{d}a_\lambda/\mathrm{d}\lambda$ est solution de (3.27), pour $\lambda = 0$, si

$$c_0(0) = c_0 \qquad \text{et} \qquad \left(\frac{\mathrm{d}c_0(\lambda)}{\mathrm{d}\lambda}\right)_{\lambda=0} = c'_i .$$

D'autre part,

$$\frac{\mathrm{d}a_\lambda(\omega)}{\mathrm{d}\lambda} = -\frac{1}{[1/c_0(\lambda) + f(\omega_+) - f(0)]^2}\cdot\left(\frac{-1}{c(\lambda)_0^2}\right)\frac{\mathrm{d}c_0(\lambda)}{\mathrm{d}\lambda},$$

d'où une solution de (3.27)

$$a^i(\omega) = c'_i\left[\frac{a(\omega)}{a(0)}\right]^2. \tag{3.28}$$

Comme pour l'équation (B.b), on peut montrer que cette solution est unique [16].

3·3.2. Résolution des équations en b'_α. – Ici encore les équations se séparent en posant

$$b'_\alpha(\omega) = b^0(\omega) + \Gamma_\alpha \omega b^1(\omega).$$

Portons dans (C.b) après avoir tenu compte de $b(\omega) = -a(\omega)$. On trouve:

$$\left\{\begin{aligned} b^0(\omega) = {} & -c'_0 + i d'_0\omega + \frac{\omega^2}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2} v_k^2\left\{\frac{a(\omega_k)a^{0*}(\omega_k)}{\omega_k - \omega - i\varepsilon} + \frac{a^*(\omega_k)a^0(\omega_k)}{\omega_k + \omega}\right\} + \\ & - \frac{\omega^2}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2} v_k^2\left\{\frac{a^*(\omega_k)b^0(\omega_k)}{\omega_k - \omega - i\varepsilon} + \frac{a(\omega_k)b^{0*}(\omega_k)}{\omega_k + \omega}\right\}, \\ b^1(\omega) = {} & d'_1 + \frac{\omega}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k} v_k^2\left\{\frac{a(\omega_k)a^{1*}(\omega_k)}{\omega_k - \omega - i\varepsilon} - \frac{a^*(\omega_k)a^1(\omega_k)}{\omega_k + \omega}\right\} + \\ & - \frac{\omega}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k} v_k^2\left\{\frac{a^*(\omega_k)b^1(\omega_k)}{\omega_k - \omega - i\varepsilon} - \frac{a(\omega_k)b^{1*}(\omega_k)}{\omega_k + \omega}\right\}. \end{aligned}\right. \tag{3.29}$$

Posons:

$$b^i(\omega) = -a^i(\omega) + b'^i(\omega) \qquad i = 0 \text{ ou } 1 \tag{3.30}$$

en utilisant (3.27), les deux équations (3.29) deviennent

$$b'^0(\omega) = id'_0\omega - \frac{\omega^2}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k^2}\, v_k^2 \left\{\frac{a^*(\omega_k)b'^0(\omega_k)}{\omega_k - \omega - i\varepsilon} + \frac{a(\omega_k)b'^{0*}(\omega_k)}{\omega_k + \omega}\right\}, \tag{3.31}$$

$$b'^1(\omega) = c'_1 + d'_1 - \frac{\omega}{\pi}\int_1^\infty \frac{k\,\mathrm{d}\omega_k}{\omega_k}\, v_k^2 \left\{\frac{a^*(\omega_k)b'^1(\omega_k)}{\omega_k - \omega - i\varepsilon} - \frac{a(\omega_k)b'^{1*}(\omega_k)}{\omega_k + \omega}\right\}, \tag{3.32}$$

dont une solution évidente est

$$b'^0(\omega) = id'_0\omega\,\frac{a(\omega)}{c_0}, \tag{3.33}$$

$$b'^1(\omega) = (c'_1 + d'_1)\,\frac{a(\omega)}{c_0}. \tag{3.34}$$

Cette solution est unique.

3'3.3. – Les expressions (3.28, 33) et la propriété $\widetilde{V}_\beta \Gamma_\beta = 0$ permettent de calculer c'_0 et d'_0 donnés par (C.(c)(d)). On calcule les intégrales des seconds membres en remarquant que

$$a^i(\infty) = c'_i + \frac{2}{\pi}\int_1^\infty \frac{kv^2}{\omega}\, 2\,\mathrm{Re}\,[a^*(\omega)a^i(\omega)]\,\mathrm{d}\omega, \tag{3.35}$$

$$a(\infty) = g'_0 = c_0 + \frac{2}{\pi}\int_1^\infty \frac{kv^2}{\omega}\,|a(\omega)|^2. \tag{3.35'}$$

On obtient alors

$$a^0(\infty) = \frac{g'^2_0}{c_k^2}\,c'_0 = 0, \qquad \frac{g'_0}{c_0}\,d'_0 = 0.$$

On a donc

$$c'_0 = d'_0 = 0. \tag{3.36}$$

Seuls sont différents de zéro $a^1(\omega)$ et $b'^1(\omega)$ donnés par (3.28, 34).

Les constantes c_1' ed d_1' se calculent de la même façon en utilisant (3.35, 35') et les valeurs de $a^1(\omega)$ et $b^1(\omega)$.

$$c_1' = \frac{c_0}{g_0^2}, \qquad c_1' + d_1' = \frac{1}{2 + g_0'/c_0}. \tag{3.37}$$

D'où finalement la solution des équations de Drell, Friedman et Zachariasen au premier ordre en g/π

$$\begin{cases} a_\alpha(\omega) = a(\omega) + \dfrac{g}{\pi} \Gamma_\alpha \omega \dfrac{a^2(\omega)}{(g_0/2\pi)^2}, \\ b_\alpha(\omega) = -a(\omega) + \dfrac{g}{\pi} \Gamma_\alpha \omega \left\{ -\dfrac{a^2(\omega)}{(g_0/2\pi)^2} + \dfrac{a(\omega)}{(g_0/2\pi)} \dfrac{1 + (g_0/2\pi) f(0)}{3 + (g_0/2\pi) f(0)} \right\}, \\ a(\omega) = \dfrac{g_0/2\pi}{1 + (g_0/2\pi) f(\omega_+)}, \quad f(\omega_+) = \dfrac{2}{\pi} \displaystyle\int_1^\infty \frac{k \omega_k \, \mathrm{d}\omega_k v_k^2}{\omega_k^2 - \omega^2 - i\varepsilon}. \end{cases} \tag{3.38}$$

Cette solution est la seule qui tende vers zéro avec g_0.

4. – Résultats numériques. Discussion.

En prenant une fonction de cut-off de la forme

$$v_k = \frac{m^2}{m^2 + k^2},$$

on obtient aisément une expression analytique pour $a_\alpha(\omega)$ et pour les tangentes des déphasages. On a:

$$f(\omega_{p+}) = \frac{m}{2} v_p (2v_p - 1) + i p v_p^2.$$

On obtient bien alors pour tg $\delta_\alpha(p)$ une fonction quasi linéaire en k aux basses énergies (jusqu'à 200 MeV pour les valeurs numériques adoptées ci-dessous).

A l'énergie nulle, on a:

$$-\left(\frac{\mathrm{tg}\, \delta_\alpha(k)}{k}\right)_{k=0} = a_\alpha(1) = \frac{g_0'}{1 + g_0'(m/2)} + \Gamma_\alpha \frac{g'}{(1 + g_0'(m/2))^2}. \tag{4.1}$$

Adoptons pour $g_0 = 2\pi g_0'$, $g = \pi g'$, les valeurs (2.3) déterminées au moyen de

l'approximation de Tomonaga, on trouve, pour $m = 4$

$$\text{(4.2)} \qquad \begin{cases} \delta_1 = -0.3777\,k\,, \\ \delta_3 = -0.3776\,k\,. \end{cases}$$

En comparant ces résultat avec les valeurs expérimentales, on voit que les deux déphasages prévus pour la diffusion par la source seule sont, d'une part, trop grands en valeur absolue et, d'autre part, beaucoup trop peu séparés

Rappelons les diverse approximations qui nous ont conduit aux résultats (4.1, 2):

1) Nous avons utilisé l'approximation à un *méson intermédiaire* dans l'approximation de la source fixe. Cela nous a permis de remplacer le véritable hamiltonien de la source fixe par le hamiltonien H_i (1.1) au plus quadratique en φ.

2) Nous avons adopté les définitions (1.3) pour les rapports de renormalization au lieu des définitions données par HORWITZ.

3) Enfin l'approximation de Tomonaga nous a servi à déterminer g_0 et g.

En admettant que la vraie valeur de g est effectivement petite, la solution des équations de Drell, Friedman et Zachariasen que nous avons trouvée est certainement très bonne.

L'approximation de Tomonaga semble devoir donner une valeur assez exacte des rapports ϱ_1 et ϱ_2, c'est-à-dire de la constante g; l'incertitude sur cette constante provient donc des approximations 1° et 2°.

La vraie valeur que l'on doit attribuer à g_0, c'est-à-dire à la masse non rénormalisée M^0, est beaucoup plus incertaine. Mais M^0 est bornée supérieurement par $27.4\,\mu$ (valeur obtenue avec le hamiltonien H_1 seul); et, par ailleurs, M^0 ne peut guère descendre en dessous de $7\,\mu$, car μ/M^0 est le paramètre de développement de l'approximation de la source fixe, qui semble être excellente, tout au moins pour l'étude de l'état P. Ainsi g_0 doit être compris entre la valeur adoptée, $10\,\mu^{-1}$ et $39\,\mu^{-1}$. Mais $a(\mu)$ est très peu sensible aux variations de g_0 pour les grandes valeurs de cette constante comme le montre l'équation (4.1) (voir Tableau II).

TABLEAU II. – *Valeur de $a(\mu)$ pour différentes valeurs de g_0 et du cut-off m.*

g_0	m		
	4	4.5	5
9.68	0.38	0.34	0.32
∞	0.50	0.44	0.40

5. – Conclusion.

La principale incertitude provient donc finalement de l'approximation à un méson intermédiaire. Si nous admettons qu'elle n'introduit qu'une erreur négligeable, il reste alors établi que les déphasages S prévus par la théorie de la diffusion par la source seule sont trop grands en valeur absolue. Le fait que nous obtenons une différence très petite entre δ_1 et δ_3 est peut-être plus fortement tributaire des autres approximations faites.

On sait que la diffusion S est principalement due à des processus dans lesquels le nucléon-source passe par des états intermédiaires d'énergie négative. Ainsi les expériences de diffusion méson-nucléon, comme par ailleurs les études de forces nucléaires, semblent montrer une faible interaction méson-nucléon dans l'état S, c'est-à-dire une faible probabilité de la formation de paires nucléon-antinucléon dans les états intermédiaires, contrairement à ce que laisserait prévoir la théorie relativiste pseudo-scalaire.

Nous nous sommes demandé si comme l'avaient déjà suggéré DYSON et MITRA [17], l'interaction méson-méson ne pourrait jouer un rôle important dans le diffusion S à basse énergie, en particulier en permettant d'expliquer cette apparente suppression des paires. Dans un prochain travail [18] nous traiterons ce problème en utilisant l'amplitude de diffusion (3.38) que nous considérerons comme représentant correctement la diffusion par la source seule.

* * *

L'idée de départ de ce travail m'a été suggéré par M. le Professeur M. LÉVY à qui va toute ma reconnaissance pour les conseils et encouragements qu'il n'a cessé de me prodiguer. Je remercie également M. J. MANDELBROJT, M. K. CHADAN et M. A. MARTIN qui m'on éclairé de leurs conseils tout particulièrement dans la résolution des équations du type de Mushkelishvili.

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[17] F. J. DYSON et A. MITRA: *Phys. Rev.*, **90**, 372 (1953).
[18] Voir également G. BONNEVAY: *Thèse Paris* (1958).

RIASSUNTO (*)

Si discutono i valori degli sfasamenti S della diffusione mesone-nucleone dedotti dalla teoria della sorgente fissa considerata come limite della teoria relativistica pseudoscalare indipendente dalla carica, in quanto la costante di accoppiamento è determinata dalla diffusione P. Le ampiezze di diffusione così ottenute sono troppo grandi ed entrambe dello stesso segno. Esse serviranno come punto di partenza per un prossimo lavoro nel quale si terrà conto della diffusione del mesone incidente da parte della nube del nucleone bersaglio.

(*) *Traduzione a cura della Redazione.*

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On the Validity of the Exponential Law for the Decay of an Unstable Particle (*).

M. LÉVY

Ecole Normale Supérieure, Université de Paris

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Summary. — The departure from the exponential decay law of an unstable particle is discussed, with the help of the Lee model, starting from an initial state which is a general superposition of eigenstates of the Hamiltonian. It is shown that the asymptotic behaviour and the magnitude of the additional terms are strongly affected by the production mechanism; although they can never vanish exactly, these terms can be made arbitrarily small by an appropriate preparation of the initial state. A possible generalization of this result is discussed.

1. – Intoduction.

Recently, several articles have been published on the mathematical definition of unstable particles in quantum mechanics or field theory ([1-3]). In some of these papers, it was pointed out that the exponential decay law for these particles is only an approximation, and that other terms are present in the asymptotic expansion of the probability $P(t-t_0)$ to observe a particle at time t, knowing that it was produced at time t_0. This property was briefly mentioned in our previous paper ([4]), where we examined the possibility of

(*) Supported in part by the United States Air Force through the European Office, Air Research and Development Command.

([1]) V. GLASER and G. KÄLLÉN: *Nucl. Phys.*, **2**, 706 (1956-57); H. ARAKI, Y. MUNAKATA, M. KAWAGUCHI and T. GOTÔ: *Prog. Theor. Phys.*, **17**, 419 (1957).

([2]) P. T. MATTHEWS and A. SALAM: *Phys. Rev.*, **112**, 283 (1958).

([3]) G. HÖHLER: *Zeits. f. Phys.*, **152**, 546 (1958); B. ZUMINO: *Lectures at the International Spring School of Physics*, Naples (1959).

([4]) M. LÉVY: *Nuovo Cimento*, **13**, 115 (1959), referred to as (I) in the following.

defining the mass and life-time of an unstable particle by locating a complex pole in the second sheet of the Riemann surface, into which its propagator can be continued analytically. The purpose of the present note is to discuss the problem of the decay curve a little more in detail. All the notations will be the same as in (I).

The existence of additional terms in $P(t-t_0)$ has no profound reasons. It is simply a consequence of the hermiticity of the Hamiltonian in quantum mechanics and of the fact that it must have only positive eigenvalues. Suppose, for example, that can we « define » (5) a state $|0\rangle$ for the particle at $t=t_0$. The probability amplitude to find the system in the same state at time t will be:

$$a(t-t_0) = \langle 0 | \exp[-iH(t-t_0)] | 0 \rangle . \tag{1}$$

The state $|0\rangle$ cannot be an eigenstate of H. To evaluate $a(t-t_0)$, we can use a complete set of such states $|n\rangle$ and write:

$$\left\{ \begin{aligned} a(t-t_0) &= \sum_n \langle 0|n\rangle\langle n|\exp[-iH(t-t_0)]|0\rangle \\ &= \sum_n \exp[-iE_n(t-t_0)]\,|\langle 0|n\rangle|^2 . \end{aligned} \right. \tag{2}$$

Since $|0\rangle$ represents an unstable particle, the states $|n\rangle$ for which $\langle 0|n\rangle$ does not vanish will in general belong to the continuous spectrum. We can therefore put $G(E_n) = \langle 0|n\rangle$ and write:

$$a(t-t_0) \simeq \frac{1}{2\pi}\int_{E_0}^{\infty} \exp[-iE(t-t_0)]\,|G(E)|^2\,dE . \tag{3}$$

The integral on the right hand side is the Fourier transform of a function which is zero for $-\infty < E < E_0$ and never negative for $E_0 \leqslant E < +\infty$. It follows (6) that $a(t)$ cannot decrease exponentially for large values of t, but behaves in general like a certain power of $1/t$. If $|G(E)|^2 \simeq g_0 (E-E_0)^p$ for $E \to E_0$, then $a(t) \simeq t^{-(p+1)}$ for large t. If $|G(E)|^2$ and all its derivatives vanish for $E \to E_0$, then $a(t)$ can behave like $\exp[-\alpha t^\nu]$, where $\nu < 1$.

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(5) The arbitrariness in such a « definition » is discussed later. In any case, such a state can be « prepared » as a superposition of eigenstates of the Hamiltonian if the details of the production mechanism are known.

(6) See, for example, S. Bochner and K. Chandrasekharan: *Fourier Transforms* (Princeton, 1949); G. Doetsch: *Theorie und Anwendung der Laplacetransformation* (Berlin, 1937).

The « state » $|0\rangle$ is not an eigenstate of H. Its definition is, to a large extent, arbitrary. *It will essentially depend on the production process.* Let us call Q the energy release in the decay and Γ the inverse half-life. In general, $1/Q$ is of the order of the production time and is much smaller than the life time τ. For $t-t_0 \gg 1/Q$, it will quite generally be possible to put $a(t-t_0)$ under the approximate form:

$$(4) \quad a(t-t_0) \simeq C\left[\exp\left[-\Gamma(t-t_0)\right] + \eta \frac{\exp\left[-i[Q(t-t_0)+\alpha]\right]}{[Q(t-t_0)]^p} + O(t^{-p-1})\right],$$

where C and η depend in an essential way on the production process. On the other hand, p is more related to the decay mechanism but can also be affected by the production process; α is an unimportant phase factor. If the state $|0\rangle$ has been normalized to unity, $|C|^2$ will be, in general, different from 1. However, since $|0\rangle$ is not observable, in most experimental conditions it will be $|C|^2$ that will be taken equal to unity, the norm of the state $|0\rangle$ being then different from 1. (These questions will be discussed with precision in Section 2). Consequently, the dominant exponential term will be the only one to be independent of the production mechanism. The coefficient η will, in general, be of the order of g^2, where g is decay coupling constant. However, it will be shown in Section 2 that η *can be made as small as one wishes if* $|0\rangle$ *is defined properly.* In other words, although $|0\rangle$ can never be an exact eigenstate of the Hamiltonian, an approximate one can be constructed to any order of accuracy (in all this discussion, g^2 is supposed to be sufficiently small so that an expansion in powers of g^2 converges very rapidly; actually, if g^2 is not very small, the notion of unstable particle itself loses its meaning).

There is another physical reason which limits the possibility of observing departures from the exponential. If we calculate $P(t-t_0)$, keeping the first two terms of (4), we have:

$$(5) \quad P(t-t_0) = |a(t-t_0)|^2 = \exp\left[-2\Gamma(t-t_0)\right] + \frac{2\eta \exp\left[-\Gamma(t-t_0)\right]}{[Q(t-t_0)]^p} \cos\left[Q(t-t_0)+\alpha\right] + \frac{\eta^2}{[Q(t-t_0)]^{2p}}.$$

In most experimental conditions, the production time t_0 is not defined. Even if it can be defined, this will never be with an accuracy comparable to $1/Q$ (which is, in most cases, of the order of 10^{-21} to 10^{-23} s). Furthermore, the time t of observation will not be defined with such a precision. Consequently, an average over $t-t_0$ will have to be made, and this will cancel the cosine term on the right hand side of (5). The departure from the exponential law will therefore be only of order η^2.

In (I), the Lee model was used extensively to study some of the properties of the propagator of an unstable particle. In that case, the model introduced a crucial ambiguity since the analytical continuation of the propagator depended on the analyticity of the cut-off function. No such ambiguity exists in the present case; the Lee model is actually very closely related to the approximation of the Heitler damping method in the quantum theory of atomic levels. This model will therefore be used in detail in Section 2 in order to study more precisely the behaviour of $P(t)$ which has been outlined in the above discussion. Usually when the initial state $|0\rangle$ is known, it is possible to express it as a sum of eigenstates of H and to determine its evolution in time. *In the case of the Lee model, it turns out that the inverse problem is also exactly soluble.* If we know the projection of the state $|0\rangle$ over the complete set of scattering states of the decay products, we can determine the state $|0\rangle$ itself by solving an integral equation of the Muskhelisvhili-Omnès type ([7,8]). We can therefore calculate exactly the initial state which will correspond to a given asymptotic form of $a(t)$.

In Section 3, a possible generalization of the decay law to a relativistic field theoretical system is discussed and compared to the work of MATTHEWS and SALAM ([9]). The general conclusion is that the additional terms to the exponential decay law are always very small. Even if, under favorable conditions and with a great improvement of the present experimental techniques, one could hope to observe such terms, they would always depend on the production mechanism and geometry, and therefore vary from one experiment to another. Furthermore, some experiments can probably be deviced which will make them arbitrarily small.

2. – Decay law of the unstable V-particle in the Lee model.

2·1. *The probability amplitude* $a(t)$. – We use the same notations as in (I), and we write, quite generally, the « state » $|0\rangle$ as:

$$|0\rangle = N|1_V, 0, 0\rangle + \frac{g}{\sqrt{\Omega}} \sum_k \frac{f(\omega)}{\sqrt{2\omega}} \varphi(\omega) |0, 1_N, 1_k\rangle \,. \tag{6}$$

The constant N coincides with the coupling constant renormalizing factor, if we adopt the prescription (2.16 a, b) of (I). $\varphi(\omega)$ is an arbitrary function which depends on the way in which the « state » $|0\rangle$ has been prepared, *i.e.* on

([7]) MUSKHELISVHILI: *Trud. Tbil. Mat. Inst.*, **10**, 1 (1941).

([8]) R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1958).

([9]) P. T. MATTHEWS and A. SALAM: *Relativistic Theory of Unstable Particles*, II (preprint).

the production mechanism. In general it will also depend on $z_0 = x - iy_0$, which is the complex pole representing the mass and life-time of the unstable V-particle. It is the root of

$$H(z_0) \equiv h(z_0) + 2i\Gamma(z_0) = 0\,, \tag{7}$$

the function $h(z)$ being defined by Eq. (2.14) of (I). If we require $|0\rangle$ to be normalized, the constant N^2 will be related to φ by the relation

$$N^2 + \frac{1}{\pi}\int_{\mu}^{\infty} \Gamma(\omega)\,|\varphi(\omega)|^2\,\mathrm{d}\omega = 1\,. \tag{8}$$

However, as was explained in the previous section, there is no physical necessity to normalize the state $|0\rangle$. Instead, another condition on N^2 can be prescribed, which corresponds more to the experimental procedure: it consists in equating to 1 the coefficient $|C|^2$ of the exponential in $P(t)$.

Since the scattering states $|N, \theta; \omega\rangle$ form a complete set, we can write the probability amplitude:

$$a(t) = \langle 0\,|\exp[-iHt]\,|0\rangle = \frac{1}{\pi}\int_{\mu}^{\infty} \Gamma(\omega)\,|U(\omega)|^2 \exp[-i(m_N + \omega)t]\,\mathrm{d}\omega\,, \tag{9}$$

with:

$$U(\omega) = \frac{\sqrt{2\omega\Omega}}{gf(\omega)}\,\langle N, \theta; \omega\,|0\rangle\,. \tag{10}$$

Using Eqs. (2.19) and (2.20) of (I), we find easily

$$U(\omega) = \varphi(\omega) - \frac{1}{h(\omega - i\varepsilon)}\,[N^2 + G(\omega - i\varepsilon)]\,, \tag{11}$$

where we have put

$$G(z) = \frac{1}{\pi}\int_{\mu}^{\infty} \Gamma(\omega)\,\varphi(\omega)\,\frac{1}{\omega - z}\,\mathrm{d}\omega\,. \tag{12}$$

Another form of $a(t)$ can be obtained by solving the time-dependent Schrödinger equation for the state $|t\rangle$ such that it reduces to $|0\rangle$ when $t \to 0$. This has been done explicitly by GLASER and KÄLLÉN in the Appendix of their paper ([1]), using Heitler's method of Laplace transforms ([10]). One finds,

([10]) W. HEITLER: *Quantum Theory of Radiation*, 2-nd ed. (Oxford, 1954).

in this way

$$(13)\qquad |t\rangle = \frac{\exp[-im_N t]}{\sqrt{\Omega}} \sum_k \frac{g f(\omega)}{\sqrt{2\omega}} \varphi(\omega) \exp[-i\omega t] |0, 1_N, 1_k\rangle -$$

$$- \frac{1}{2\pi i} \exp[-im_N t] \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \exp[-izt]\, dz\, \frac{[N^2 + G(z)]}{h(z)} \cdot$$

$$\cdot \left[N |1_V, 0, 0\rangle + \frac{g}{\sqrt{\Omega}} \sum_k \frac{f(\omega)}{\sqrt{2\omega}} \frac{1}{\omega - z} |0, 1_N, 1_k\rangle \right].$$

Consequently, we find for $a(t)$

$$(14)\qquad a(t) = \langle 0 | t\rangle = \frac{1}{\pi} \int_\mu^\infty \Gamma(\omega) |\varphi(\omega)|^2 \exp[-i(m_N + \omega)t]\, d\omega -$$

$$- \frac{1}{2\pi i} \exp[-im_N t] \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \exp[-izt] \frac{dz}{h(z)} [N^2 + G(z)][N^2 + \bar{G}(z)],$$

where $\bar{G}$ is defined by

$$(15)\qquad G(z) = \frac{1}{\pi} \int_\mu^\infty \Gamma(\omega) \frac{\varphi^*(\omega)}{\omega - z}\, d\omega .$$

Eq. (14) is useful to obtain the exponential term of $a(t)$: for this purpose, it is sufficient to change the contour of integration slightly, so as to enclose the pole z_0 in the second sheet of the Riemann surface constructed from the cut $\mu \leqslant \omega \leqslant \infty$. Eq. (9), on the other hand, can be used to calculate the asymptotic form of $a(t)$. If $|U(\mu)|^2 \neq 0$, we have

$$(16)\qquad a(t) \simeq \frac{1}{2\sqrt{\pi}} \frac{g^2}{4\pi} |f(\mu)|^2 \sqrt{2\mu}\, |U(\mu)|^2 \exp\left[-i\left(m_N + \mu + \frac{3\pi}{4} \right) t \right] t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}) .$$

For $\omega = \mu$, the functions $h(\omega)$ and $G(\omega)$ which appear in (11) and (12) are real and well defined. It is therefore clear that a real function $\varphi(\omega)$ (it does not need, actually, to depend explicitly on z_0 in this case) can be chosen in such a way that $U(\omega)$ and an arbitrary number of its derivatives vanish when $\omega \to \mu$. This means that a particular choice of the initial « state » $|0\rangle$ will strongly affect the asymptotic behaviour of $a(t)$, since the leading power in the expansion in $1/t$ can be chosen arbitrarily.

2·2. *Determination of the initial state for an arbitrary* $U(\omega)$. – In general, $U(\omega)$ (defined by Eq. (10)) will be a function which will be small everywhere

except for a narrow peak around $\omega = x_0$. If we want $|0\rangle$ to be approximately an eigenstate of H, we have to make $U(\omega)$ as close to a δ-function as possible. Note that, if the « state » $|0\rangle$ is normalized to unity, we have

$$\frac{1}{\pi}\int_{\mu}^{\infty} \Gamma(\omega)\,|U(\omega)|^2\,\mathrm{d}\omega = 1\,. \tag{17}$$

$U(\omega)$ is proportional to the projection of the « state » $|0\rangle$ over the scattering state $|N, \theta; \omega\rangle$. If it is given, we can determine $\varphi(\omega)$, and therefore the « state » $|0\rangle$ itself, by solving the integral equation deduced from Eqs. (11) and (12). The forward scattering amplitude $f_+(\omega) = \exp[i\delta_0(\omega)] \sin\delta_0(\omega)$ is given by Eq. (2.21) of (I). Writing it as

$$-f_+(\omega) \equiv K^*(\omega) = \exp[-i\delta]\sin\delta = \frac{\Gamma(\omega)}{h(\omega + i\varepsilon)}\,, \tag{18}$$

where $\delta = -\delta_0$ is the opposite of the scattering phase-shift, and putting

$$\lambda(\omega) = 1 + \frac{h(\omega + i\varepsilon)}{N^2}\,U^*(\omega)\,, \tag{19}$$

$$\psi(\omega) = \frac{h(\omega + i\varepsilon)}{N^2}\,\varphi^*(\omega)\,, \tag{20}$$

we can write Eqs. (11) and (12) in the form

$$\psi(\omega) = \lambda(\omega) + \frac{1}{\pi}\int_{\mu}^{\infty} \frac{K^*(\omega)\psi(\omega')}{\omega' - \omega - i\varepsilon}\,. \tag{21}$$

This integral equation for $\psi(\omega)$ is of the Muskhelisvhili-Omnès ([7,8]) type. Its solution has now become classical, so that we shall only quote the result, in the simplified form given by Gourdin and Martin ([11]). We define:

$$u(z) = \frac{1}{\pi}\int_{\mu}^{\infty} \frac{\delta(\omega)\,\mathrm{d}\omega}{\omega - z}\,. \tag{22}$$

The solution of (21) is then given by

$$\psi(\omega) = \lambda(\omega) + \exp[u(\omega + i\varepsilon)]\left[\lambda(\infty) - \frac{1}{2\pi i}\int_{C} \frac{\lambda(z)\exp[-u(z)]}{z - \omega - i\varepsilon}\right], \tag{23}$$

([11]) M. Gourdin and A. Martin: *Nuovo Cimento*, **8**, 699 (1958).

where the contour C of integration is drawn on Fig. 1. It should be noted that, if $\lambda(z)$ does not have any pole inside the contour, Eq. (23) reduces to

$$\psi(\omega) = \exp[u_+(\omega)]\lambda(\infty)\,, \tag{24}$$

where we put

$$u_\pm(\omega) = u(\omega \pm i\varepsilon) = \varrho(\omega) \pm i\,\delta(\omega)\,. \tag{25}$$

The function $\exp[u(z)]$ is related, in all cases, to the function $h(z)$ defined by (2.14) of (I) through the relation

$$h(z) = \exp[u(z)](N^2 z - \beta)\,, \tag{26}$$

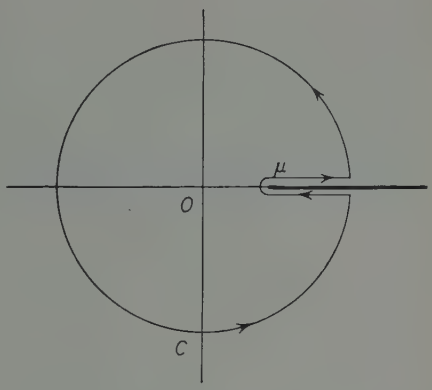

Fig. 1.

where β is a certain constant which is given explicitly for the different possible forms of the mass spectrum in the appendix. In the case where the V-particle is unstable, a convenient form is given by

$$h_+(\omega) = \exp[u_+(\omega)]\,[N^2(\omega - \alpha) - i\Gamma(\alpha)\exp[-u_-(\alpha)]]\,, \tag{27}$$

where α is defined by

$$\overline{h}(\alpha) = \mathrm{Re}\,(h_\pm(\alpha)) = 0\,. \tag{28}$$

In the case of an unstable V-particle, it should be noted also that the homogeneous integral equation ($\lambda = 0$) does not have any other solution than $\psi = 0$. For $U(\omega) = 0$, we have $\lambda = 1$, and the integral equation has the solution

$$\psi = \exp[u_+(\omega)]\,. \tag{29}$$

In that case, it is easy to verify that the « state » $|0\rangle$ has a zero norm, as should be expected, since it is orthogonal to all the (N, θ) scattering states which already form a complete set. This will also be true for any $U^*(\omega)$ of the form

$$U^*(\omega) = \frac{g(\omega)}{h(\omega + i\varepsilon)}\,, \tag{30}$$

where $g(\omega)$ is a regular function inside the contour C.

In order to prove that the coefficient η of Eq. (4) can be made as small as we like, we shall select a particular form of $U^*(\omega)$ of the type

$$U_n^*(\omega) = \frac{C_n}{h(\omega + i\varepsilon)(\omega - z_0)^n} . \tag{31}$$

If n is large enough, this corresponds to a distribution $|U_n(\omega)|^2$ which has a very narrow peak around $\omega = x_0$, much more narrow, in any case, than the distribution given by the ϱ-function of Lehmann ([12]) (which corresponds to $n = 0$). The coefficient C_n is defined by Eq. (17), which gives, in this case,

$$\frac{1}{C_n^2} = \frac{1}{\pi}\int_{\mu}^{\infty} \frac{\Gamma(\omega)\,\mathrm{d}\omega}{|h_+(\omega)|^2\,|\omega - z_0|^{2n}} = -\frac{1}{2\pi i}\int_{\sigma} \frac{\mathrm{d}z}{h(z)[(z - x_0)^2 + y_0^2]^n} . \tag{32}$$

The integral on the right hand side is the sum of the two residues at $z = z_0$ and $z = z_0^*$. If $z_0 - z_0^* = -2iy_0$ is sufficiently small, we can evaluate it approximately by

$$\frac{1}{C_n^2} = -\frac{1}{(n-1)!}\left\{\frac{1}{h(z_0)}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(z - z_0^*)^{-n}\right]_{z=z_0} + \frac{1}{h(z_0^*)}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}(z - z_0)^{-n}\right]_{z=z_0^*}\right\} . \tag{33}$$

Using the relation

$$h(z_0) - h(z_0^*) = -4i\,\mathrm{Re}\,\Gamma(z_0) \simeq -4i\Gamma(x_0) , \tag{34}$$

we have approximately,

$$\left[\frac{C_n}{(n-1)!}\right]^2 = \frac{\Gamma(x_0)(2y_0)^{2n-1}}{(2n-2)!} ; \tag{35}$$

we see that C_n is of the order of g^{2n}, where g is the decay coupling constant (y_0 is of the order of g^2).

With the form (31) of $U_n^*(\omega)$, we have

$$\lambda(\omega) = 1 + \frac{C_n}{N^2(\omega - z_0)^n} , \tag{36}$$

and the solution is then given by

$$\psi^*(\omega) = \exp[u_+(\omega)]\left[1 - \frac{C_n}{N^2(n-1)!}\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\frac{\exp[-u(z)]}{z - \omega}\right)_{z=z_0}\right] . \tag{37}$$

([12]) H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

The corresponding value of $|U(\mu)|^2$ is

$$|U(\mu)|^2 = \frac{C_n^2}{|h_+(\mu)|^2} \frac{1}{|(z_0 - \mu)|^{2n}} \simeq \frac{C_n^2}{|h_+(\mu)|^2 Q^{2n}}, \tag{38}$$

where $Q = x_0 - \mu = m_V - m_N - \mu$. Consequently, the coefficient η of Eq. (4) is of the order of $(g)^{4n+2}$. If the decay is long enough, it can be made as small as one wishes by selecting a sufficiently high value of n, and, consequently, an appropriate form for $\varphi(\omega)$ and for the « state » $|0\rangle$.

As was said in Section **1**, this result can easily be generalized to the case of a more complicated Hamiltonian than the one of the Lee model, if one makes use of the « damping theory » of Heitler ([10]).

3. – Possible generalizations of the decay time-plot.

The difficulties of defining a « state » for an unstable particle in a general field theory have already been emphasized in (I). One might hope that an unstable particle can be represented by a field $A(x)$ which can perhaps be constructed out of the fields of the stable decay products by a generalization of the method proposed by ZIMMERMANN ([13]). However, $A_{in}(x)$ and $A_{out}(x)$, which are the limits of $A(x)$ when $t \to \mp \infty$ do not exist and can therefore not be used to construct a basis of asymptotic states.

In their recent paper ([9]), MATTHEWS and SALAM have proposed a possible definition for the relativistic state vector which is a linear superposition of the decaying states, weighted by the ϱ-function of Lehmann ([12]). This yields, for the particle considered in its rest-system,

$$a_j(t) = \int \varrho_j(K^2) \exp[-iKt] \mathrm{d}K^2, \tag{39}$$

where $\varrho_j(K^2)$ is the partial ϱ-function defined by

$$\varrho_j(K^2) = 2\pi \sum_j |\langle 0|A(x)|j\rangle|^2 \delta(k_j^2 + K^2), \tag{40}$$

where the sum over the states $|j\rangle$ is limited to those into which the particle actually decays. The state vector proposed by MATTHEWS and SALAM is admittedly to a large extent arbitrary, since the production process should again manifest itself in the way in which the « state » of the unstable particle is prepared. The intuitive picture which the above authors use is that of a

([13]) W. ZIMMERMANN: *Nuovo Cimento*, **8**, 597 (1958).

superposition of plane waves corresponding to various masses of the unstable particle, weighted just by Lehmann's function. However, there is no *a priori* reason to have this mass weight function coincide with the ϱ-function. In the previous section, we have seen that, in general, the mass weight function has the form [14]

$$\frac{1}{\pi}\Gamma(\omega)\,|\,U(\omega)\,|^2 = \varrho(\omega)\,|\,T(\omega)\,|^2\,, \tag{41}$$

where $T(\omega)$ is a certain function more or less peaked around the value x_0. In Section 2 we have shown, in fact, that the mass weight-function in the decay time-plot can be much more narrowly peaked around the average mass than the ϱ-function itself, altough it can never, of course, reach the limit of a δ-function. In general, one can guess that the probability amplitude will have the form

$$a(t) = \int \varrho_j(K^2)\,|\,T(K^2)\,|^2 \exp\left[-iKt\right] \mathrm{d}K^2\,, \tag{42}$$

where the T-function, which should be introduced as a coefficient in the linear superposition of Matthews and Salam (Eq. (4.1) of their paper), reflects the arbitrariness in the definition of the initial state. On the other hand, it is true that one cannot take the complete Lehmann's function ϱ but only the partial ϱ_j defined by (40). (This can be verified, for example, with the help of the « multiple Lee model » introduced in our previous paper [15]). Otherwise, there would be a large probability for the Λ-particle, for example, to decay into p+K, even though the decay coupling constant is very small.

* * *

The author has greatly benefited, in the development of the ideas contained in the present paper, from the stimulating discussions which took place at the International Spring School of Physics, in Naples (April 1959). He would

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[14] Lehmann's function $\varrho(\omega)$, in the case of the Lee model, has been given in (I), Eq. (3.4).

[15] In this model, the V and N particles are coupled to a series of $n+1$ bosons θ^p ($p = 0, 1, ..., n$) of *increasing* masses μ_p in such a way that $m_N+\mu_0 < m_V < m_N+\mu_1$. The corresponding coupling constants g_p are such that $g_0 \ll g_i$ ($i \neq 0$). In a first approximation, it is natural to neglect g_0 and to calculate the *physical state* of the stable V-particle in the absence of the weak decay interaction. The latter is then introduced, the initial state being the physical V-particle state vector calculated previously. This simple calculation enables one to verify the statements made in this Section, and to calculate the T-function explicitly. As was already mentioned in (I), this model reproduces many of the features of a more general theory where the number of mesons is not limited, especially if the masses μ_p are chosen in such a way that $\mu_p = \mu p$ ($p \neq 0$).

like to thank especially Prof. E. R. CAIANIELLO for his kind hospitality. He is also indebted to Dr. M. GOURDIN for a stimulating discussion on the solution of the Omnès-Muskhelisvhili integral equation.

APPENDIX

Using Eq. (2.14) of (I), we can write

$$h_+(\omega) - h(z_1) = (\omega - z_1)\left[N^2 + \frac{1}{\pi}\int_\mu^\infty \frac{\Gamma(\omega')\,\mathrm{d}\omega'}{(\omega' - z_1)(\omega' - \omega - i\varepsilon)}\right], \tag{A.1}$$

where z_1 is any point of the complex plane outside the cut $(+\mu, +\infty)$. Using Eq. (18) we can write:

$$\frac{h_+(\omega)}{\omega - z_1} = N^2 + \frac{h(z_1)}{\omega - z_1} + \frac{1}{\pi}\int_\mu^\infty \frac{h_+(\omega')}{\omega' - z_1}\cdot\frac{K^*(\omega')\,\mathrm{d}\omega'}{\omega' - \omega - i\varepsilon}, \tag{A.2}$$

so that $(\omega - z_1)^{-1}h_+(\omega)$ is a solution of an Omnès equation with

$$\lambda(\omega) = N^2 + \frac{h(z_1)}{\omega - z_1}. \tag{A.3}$$

Using Eq. (23), we see, first of all, that

$$\frac{1}{2\pi i}\int_\sigma \frac{\lambda(z)\exp[-u(z)]}{z - \omega - i\varepsilon} = \lambda(\omega)\exp[-u_+(\omega)] - \frac{h(z_1)}{\omega - z_1}\exp[-u(z_1)], \tag{A.4}$$

and

$$\frac{h^+(\omega)}{\omega - z_1} = \exp[u_+(\omega)]\left[N^2 + \frac{h(z_1)}{\omega - z_1}\exp[-u(z_1)]\right]. \tag{A.5}$$

a) In the case when a stable V-particle exists, $h(\omega)$ has a real root for $\omega = \omega_0 < \mu$. We can therefore take $z_1 = \omega_0$ and write

$$h_+(\omega) = N^2\exp[u_+(\omega)](\omega - \omega_0). \tag{A.6}$$

b) If there is no stable V-particle, we can take either $z_1 = z_0$, which, taking into account Eq. (7), gives

$$h_+(\omega) = \exp[u_+(\omega)]\{N^2(\omega - z_0) - 2i\Gamma(z_0)\exp[-u(z_0)]\}, \tag{A.7}$$

or, take $z_1 = \alpha - i\varepsilon$ where α is such that

$$\text{(A.8)} \qquad \text{Re}\, h_{\pm}(\alpha) \equiv \bar{h}(\alpha) = 0 \,.$$

For $g^2/4\pi \ll 1$, α coincides practically with x_0. This gives then Eq. (41) of the text.

Note added in proof.

1) In connection with the discussion of Sect. 2·2, it is worth while to mention that a simple choice of $\varphi(\omega)$, namely:

$$\varphi(\omega) = \text{Re}\,\frac{1}{\omega - z_0} \,,$$

already gives a value of η, in Eq. (4), of the order of g^{10}! ...

2) The discussion of the form of $h_+(\omega)$ in the Appendix is not complete in the case *b*). In general, a solution of the homogeneous equation corresponding to (A.2) must be added on the right hand side of (A.5). In the case *a*), it turns out that $\delta(0) - \delta(\infty) = 0$, so that the homogeneous equation has no solution. In the case *b*), where the phase-shift goes through $\pi/2$, the homogeneous equation has a non vanishing solution (see the discussion of ref. [11]). In any case, the result is clear: one must have

$$\bar{h}(\omega) = N^2 \exp\,[\varphi(\omega)](\omega - \alpha) \,,$$

where

$$\varphi(\omega) = \frac{1}{\pi} P \int_{\mu}^{\infty} \frac{\delta(\omega')\,\mathrm{d}\omega'}{\omega' - \omega} \,.$$

RIASSUNTO (*)

Con l'ausilio del modello di Lee si discute lo scostamento del decadimento di una particella instabile dalla legge esponenziale, partendo da uno stato iniziale risultante da una sovrapposizione generale di autostati dell'hamiltoniana. Si dimostra che il comportamento asintotico e la grandezza dei termini aggiuntivi sono fortemente influenzati dal meccanismo di produzione; per quanto non possano mai esattamente annullarsi, questi termini si possono rendere arbitrariamente piccoli con una costruzione appropriata dello stato iniziale. Si discute una possibile generalizzazione di tale risultato.

(*) *Traduzione a cura della Redazione.*

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Etude critique et forme simplifiée résoluble de l'approximation de couplage intermédiaire de Tomonaga.

J. Mandelbrojt

Ecole Normale Supérieure - Paris

(ricevuto il 22 Luglio 1959)

Summary. — Tomonaga's intermediate coupling approximation is discussed critically for all values of the coupling constant. A simplified approximation which has similar properties is given and the corresponding equations are solved explicitly.

Introduction.

L'approximation de couplage intermédiaire de Tomonaga [1] donne par un calcul variationnel une expression approchée de l'état fondamental d'un nucléon entouré d'un nuage de mésons virtuels. La justification que l'on donne habituellement de cette méthode est qu'elle redonne les résultats de la méthode des perturbations lorsque celle-ci est justifiée, c'est-à-dire lorsque la constante de couplage f des mésons avec le nucléon tend vers 0, et qu'elle redonne les résultats du couplage fort lorsque $f \to \infty$. Pour les valeurs « intermédiaires » de la constante de couplage, elle substitue au problème, de seconde quantification, de la recherche de l'état fondamental du nucléon, un problème plus simple et souvent résoluble, du moins numériquement, de première quantification.

Nous étudions, dans cet article, la validité de l'approximation de couplage intermédiaire pour une valeur quelconque de la constante de couplage. Ceci nous permet d'indiquer une condition nécessaire et suffisante pour que l'approximation de Tomonaga donne exactement l'état fondamental du nucléon, ce qui nous conduit à une approximation simplifiée de couplage intermédiaire

(1) S. Tomonaga: *Prog. Theor. Phys.*, **1**, 109 (1946).

donnant explicitement un vecteur d'état simple qui approche le vecteur d'état fondamental du nucléon. Nous étudions la validité de notre approximation, et nous montrons que, lorsque $f \to 0$, notre approximation, comme l'approxmation de couplage intermédiaire, donne exactement l'état fondamental du nucléon. Lorsque la constante de couplage $f \to \infty$ notre approximation se rapproche de l'approximation de couplage intermédiaire plus rapidement que celle-ci ne se rapproche du vecteur d'état fondamental vrai du nucléon. Nous traitons ici le cas du nucléon entouré par des mésons chargés qui interagissent avec lui par un couplage scalaire symétrique.

1. – Approximation de couplage intermédiaire.

L'Hamiltonien de source fixe représentant des mésons chargés agissant sur un nucléon par un couplage scalaire symétrique est

$$H = \frac{1}{(2\pi)^3}\int\{\omega(k)a_i^*(k)a_i(k) - fR(k)\tau_i(a_i^*(k) + a_i(k))\}\,\mathrm{d}\boldsymbol{k}\,. \tag{1}$$

$a_i^*(k)$ et $a_i(k)$ sont les opérateurs de création et d'annihilation d'un méson de moment $\boldsymbol{k}$ et d'indice de spin isotopique i ($i = 1, 2, 3$), $\omega(k)$ son énergie totale, $R(k)$ est la fonction de source, à symétrie sphérique, dans l'espace des moments. La méthode de couplage intermédiaire consiste à chercher une approximation de l'état fondamental du nucléon (ou en général d'un état propre de H) sous la forme:

$$|\psi_T\rangle = \sum \alpha_{P_1P_2P_3}(a_1^*)^{P_1}(a_2^*)^{P_2}(a_3^*)^{P_3}|\psi_0\rangle\,, \tag{2}$$

$|\psi_0\rangle$ étant le vecteur d'état du nucléon nu, et

$$a_i^* = \frac{1}{(2\pi)^3}\int a_i^*(k)\Phi(k)\,\mathrm{d}\boldsymbol{k}\,,$$

$\Phi(k)$ étant une fonction normée de $\boldsymbol{k}$, $((1/2\pi^3)\int\Phi^2(k)\,\mathrm{d}\boldsymbol{k} = 1)$.

C'est-à-dire que l'on suppose que l'état fondamental du nucléon est constitué par le nucléon habillé par des mésons ayant tous la même distribution angulaire $\Phi(k)$.

On peut alors démontrer facilement que

$$a_i(k)|\psi_T\rangle = \Phi(k)a_i|\psi_T\rangle\,, \tag{3}$$

et que pour deux états $|\psi_T\rangle$ et $|\psi'_T\rangle$ de la forme (2) et relatifs à la même fonction $\Phi(k)$, on a:

$$\langle \psi'^{*}_T H \psi_T \rangle = \langle \psi'^{*}_T H_T \psi_T \rangle \,, \tag{4}$$

où

$$H_T = \omega_0 a_i^* a_i - f_0 \tau_i (a_i^* + a_i) \,.$$

avec

$$\begin{cases} \omega_0 = \dfrac{1}{(2\pi)^3} \int \omega(k) \Phi^2(k) \, \mathrm{d}\boldsymbol{k} \,, \\ f_0 = \dfrac{f}{(2\pi)^3} \int R(k) \Phi(k) \, \mathrm{d}\boldsymbol{k} \,. \end{cases}$$

La recherche de l'état fondamental du nucléon, c'est-à-dire du vecteur d'état qui donne son minimum à la valeur moyenne de H, se ramène alors, d'une part, à la recherche du minimum $E_T(\Phi)$ de $\langle \psi_T^* H_T \psi_T \rangle$ (recherche des coefficients $\alpha_{P_1 P_2 P_3}$), qui est un problème de première quantification puisque H_T est formé uniquement avec les opérateurs de création et d'annihilation a_i^* et a_i, d'autre part à la recherche de la fonction $\Phi(k)$ qui rend minimum la fonctionnelle $E_T(\Phi)$. On peut ramener le premier problème à la recherche de deux oscillateurs harmoniques couplés. Nous avons indiqué dans une note aux *Comptes Rendus* une généralisation de la méthode de W.K.B. permettant une résolution approchée de ce problème [2]. Le second problème se résoud aisément.

2. – Inégalité de Weinstein.

Une condition nécessaire et suffisante pour qu'un vecteur d'état $|\psi\rangle$ soit vecteur propre d'un Hamiltonien H est que

$$\sigma^2(H, \psi) = \langle \psi^* H^2 \psi \rangle - \langle \psi^* H \psi \rangle^2 = 0 \,.$$

On peut voir ceci en notant que

$$\langle \psi^* \psi \rangle \langle \psi^* H | H \psi \rangle - \langle \psi^* H | \psi \rangle^2 \geqslant 0 \,,$$

d'après l'inégalité de Schwarz, et une condition nécessaire et suffisante pour que l'égalité ait lieu est que les vecteurs $|\psi\rangle$ et $|H\psi\rangle$ soient colinéaires, c'est-à-dire que $|\psi\rangle$ soit vecteur propre de H.

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[2] J. MANDELBROJT: *Compte-Rendu Acad. Sci.*, **247**, 871, (1958).

Nous pouvons interpréter l'expression

$$\sigma^2(H, \psi)\,,$$

lorsqu'elle n'est pas nulle, en posant

$$|H\psi\rangle = \alpha|\psi\rangle + \beta|\psi''\rangle\,,$$

$|\psi''\rangle$ étant un vecteur d'état normé orthogonal à $|\psi\rangle$. Alors $\sigma^2(H, \psi) = \beta^2$, c'est-à-dire que

$$\frac{\langle\psi^* H^2\psi\rangle - \langle\psi^* H\psi\rangle^2}{\langle\psi^* H\psi\rangle^2} = \frac{\beta^2}{\alpha^2} = \operatorname{tg}^2\gamma\,,$$

γ étant l'angle que fait le vecteur $|H\psi\rangle$ avec le vecteur $|\psi\rangle$. γ donne une indication sur l'approximation avec laquelle un vecteur d'état $|\psi\rangle$ peut être considéré comme un état propre de H.

Une interprétation utile de $\sigma^2(H, \psi)$ a été donnée par WEINSTEIN [3] qui a introduit cette expression: Si le système des fonctions propres d'un hamiltonien H est complet et si nous considérons une fonction $|\psi\rangle$ normée arbitraire, il existe une valeur propre E de H dans l'intervalle $\langle\psi^* H\psi\rangle - \sigma(H, \psi)$, $\langle\psi^* H\psi\rangle + \sigma(H, \psi)$.

Remarquons que cette évaluation de l'erreur sur l'énergie est plutôt pessimiste. En effet, developpons $|\psi\rangle$ suivant les états propres de H,

$$|\psi\rangle = \sum_n c_n|\psi_n\rangle\,, \qquad \text{avec} \qquad H|\psi_n\rangle = E_n|\psi_n\rangle\,,$$

$$\sigma^2(H, \psi) = \sum_n c_n^2 E_n^2 - \Big(\sum_n c_n^2 E_n\Big)^2\,.$$

Si la fonction d'essai $|\psi\rangle$ pour l'état fondamental, contient (à tort) des composantes $|\psi_n\rangle$ d'énergie élevée, la série $\sum c_n^2 E_n^2$ peut avoir une somme élevée et même diverger, alors que la valeur de l'énergie $\sum c_n^2 E_n$ peut n'être pas trop mauvaise.

3. – Expression de $\sigma^2(H, \psi_T)$ et condition nécessaire et suffisante pour que $\sigma^2(H, \psi_T) = 0$.

En utilisant la relation (3) et la relation transposée, nous pouvons exprimer $\sigma^2(H, \psi_T)$ avec des valeurs moyennes d'opérateurs formés uniquement avec a_i^*

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[3] D. M. WEINSTEIN: *Proc. Nat. Acad. Sc.*, **20**, 529 (1934).

et a_i. Nous supposons provisoirement que $|\psi_T\rangle$ est vecteur propre de H_T (*). $\langle\psi_T^* H\psi_T\rangle = \langle\psi_T^* H_T\psi_T\rangle$, quantité que nous désignons par E_T.

Calculons maintenant $\langle\psi_T^* H^2\psi_T\rangle$, c'est-à-dire

$$\langle\psi_T^* H, \frac{1}{(2\pi)^3}\int\{\omega(k)a_i^*(k)a_i(k) - fR(k)\tau_i(a_i^*(k) + a_i(k))\}\mathrm{d}\boldsymbol{k}\psi_T\rangle\,.$$

Nous commençons par faire les commutations nécessaires pour faire opérer directement $a_i(k)$ sur $|\psi_T\rangle$ et $a_i^*(k)$ sur $\langle\psi_T^*|$.

En utilisant la relation

$$Ha_i^*(k) = a_i^*(k)H + \omega(k)a_i^*(k) - fR(k)\tau_i\,,$$

il vient:

$$\begin{aligned}\langle\psi_T^* H^2\psi_T\rangle = \frac{1}{(2\pi)^3}\int\{&\langle\psi_T^*\omega(k)a_i^*(k)Ha_i(k)\psi_T\rangle + \langle\psi_T^*\omega^2(k)a_i^*(k)a_i(k)\psi_T\rangle - \\ &- f\langle\psi_T^*\omega(k)R(k)\tau_i a_i^*(k)\psi_T\rangle - \langle\psi_T^* HfR(k)\tau_i a_i(k)\psi_T\rangle - \\ &- \langle\psi_T^* a_i^*(k)HfR(k)\tau_i\psi_T\rangle - f\langle\psi_T^*\omega(k)R(k)\tau_i a_i^*(k)\psi_T\rangle + f^2\langle\psi_T^* R^2(k)\tau_i^2\psi_T\rangle\}\mathrm{d}\boldsymbol{k}\,.\end{aligned}$$

Nous pouvons maintenant remplacer $a_i(k)|\psi_T\rangle$ par $a_i\Phi(k)|\psi_T\rangle$ et $\langle\psi_T^*|a_i^*(k)$ par $\langle\psi_T^*|a_i^*\Phi(k)$, et, par conséquent, l'hamiltonien H au second membre par H_T, d'après (4). Nous effectuons alors les commutations nécessaires pour faire agir l'opérateur H_T directement sur $|\psi_T\rangle$ afin d'utiliser $H_T|\psi_T\rangle = E_T|\psi_T\rangle$. En utilisant la relation de commutation $a_i^* H_T = H_T a_i^* - \omega_0 a_i^* + f_0\tau_i$, il vient,

$$\frac{1}{(2\pi)^3}\int\langle\psi_T^* a_i^*\Phi(k)H_T a_i\Phi(k)\psi_T\rangle = E_T\omega_0 N - \omega_0^2 N - f_0\omega_0 M\,,$$

et

$$\frac{1}{(2\pi)^3}\int\langle\psi_T^* a_i^*\Phi(k)H_T fR(k)\tau_i\psi_T\rangle = E_T f_0 M - \omega_0 f_0 M + 3f_0^2\,,$$

où l'on a posé

$$N = \langle\psi_T^* a_i^* a_i\psi_T\rangle\,,$$

qui est le nombre moyen de mésons présents dans la nuage du nucléon,

$$M = \langle\psi_T^*\tau_i a_i\psi_T\rangle\,,$$

(*) Si $|\psi_T\rangle$ n'était pas vecteur propre de H_T, $\sigma^2(H, \psi_T)$ serait augmenté de la quantité β'^2, où β' est défini par $|H_T\psi_T\rangle = \alpha'|\psi_T\rangle + \beta'|\psi_T'\rangle$, $|\psi_T'\rangle$ étant un vecteur d'état orthogonal à $|\psi_T\rangle$.

d'où finalement:

$$\sigma^2(H, \psi_T) = \frac{1}{(2\pi)^3}\left\{N\int \omega^2(k)\Phi^2(k)\,\mathrm{d}\boldsymbol{k} - 2fM\int \omega(k)\Phi(k)R(k)\,\mathrm{d}\boldsymbol{k} + \right.$$
$$\left. + 3f^2\int R^2(k)\,\mathrm{d}\boldsymbol{k}\right\} - \{N\omega_0^2 - 2f_0M\omega_0 + 3f_0^2\},$$

que l'on peut encore récrire

$$\sigma^2(H, \psi_T) = \frac{1}{(2\pi)^3}\int \langle \psi_T^*(\omega(k)a_i^*\Phi(k) - \tau_i fR(k))(\omega(k)a_i\Phi(k) - \tau_i fR(k))\psi_T\rangle\,\mathrm{d}\boldsymbol{k} -$$
$$- \frac{1}{(2\pi)^6}\langle \psi_T^*\left\{\int \omega(k)a_i^*\Phi^2(k) - \tau_i fR(k)\Phi(k)\right\}\mathrm{d}\boldsymbol{k}\cdot$$
$$\cdot\left\{\int \omega(k')a_i^*\Phi^2(k') - \tau_i fR(k')\Phi(k')\right\}\mathrm{d}\boldsymbol{k}'\psi_T\rangle .$$

L'étude du second membre en tant qu'intégrales portant sur des fonctions de $\boldsymbol{k}$, donne d'après la condition pour que l'inégalité de Schwarz soit transformée en égalité, une condition nécessaire et suffisante pour que $\sigma^2(H, \psi_T)$ soit nulle, qui est:

$$(\omega(k)a_i\Phi(k) - \tau_i fR(k))\,|\psi_T\rangle = |A\rangle\Phi(k)\,, \tag{6}$$

$|A\rangle$ ne dépendant pas de $\boldsymbol{k}$.

En faisant le produit scalaire des deux membres de l'égalité (6) par le vecteur d'état $\langle\psi_T^*|a_i^*$ on obtient,

$$\omega(k)N\Phi(k) - MfR(k) = \lambda\Phi(k)\,, \tag{7}$$

où λ est une constante, d'où

$$\Phi(k) = \frac{fMR(k)}{N\omega(k) - \lambda}\,.$$

Ceci est une condition nécessaire pour que (5) soit nul; c'est l'expression de $\Phi(k)$ que l'on obtient dans la méthode habituelle de couplage intermédiaire en écrivant que la variation par rapport à Φ de $\langle\psi_T^*|\ H_T - E_T|\psi_T\rangle$ est nulle. L'expression de λ obtenue en intégrant par rapport à k les deux membres de (7), après les avoir multiplié par $\Phi(k)$, est $\lambda = \omega_0 N - f_0 M$.

Il est clair que l'égalité (7) n'épuise pas le sens de l'égalité (6). Nous allons maintenant chercher l'expression que doit avoir $|\psi_T\rangle$ pour que $\sigma^2(H, \psi_T) = 0$.

Un regroupement des termes de (5) donne:

$$(5') \qquad \sigma^2(H, \psi_T) = \left(N - \frac{M^2}{3}\right)\left\{\frac{1}{(2\pi)^3}\int \omega^2(k)\Phi^2(k)\,\mathrm{d}\boldsymbol{k} - \omega_0^2\right\} + \\ + \frac{1}{(2\pi)^3}\int\left(\frac{M}{3^{\frac{1}{2}}}\,\omega(k)\Phi(k) - 3^{\frac{1}{2}}fR(k)\right)^2 \mathrm{d}\boldsymbol{k} - \left(\frac{M}{3^{\frac{1}{2}}}\,\omega_0 - 3^{\frac{1}{2}}f_0\right)^2,$$

et d'après l'inégalité de Schwarz:

$$(8) \qquad \langle \psi_T^* \tau_i a_i \psi_T\rangle^2 \leqslant \langle \psi_T^* \tau_i^2 \psi_T\rangle \langle \psi_T^* a_i^* a_i \psi_T\rangle\,,$$

c'est-à-dire $N \geqslant M^2/3$, l'inégalité de Schwarz indique également que

$$(9) \qquad \frac{1}{(2\pi)^3}\int \omega^2(k)\Phi^2(k)\,\mathrm{d}\boldsymbol{k} - \omega_0^2 > 0\,,$$

et

$$(10) \qquad \frac{1}{(2\pi)^3}\int\left(\frac{M}{3^{\frac{1}{2}}}\,\omega(k)\Phi(k) - 3^{\frac{1}{2}}fR(k)\right)^2 \mathrm{d}\boldsymbol{k} - \left(\frac{M}{3^{\frac{1}{2}}}\,\omega_0 - 3^{\frac{1}{2}}f_0\right)^2 \geqslant 0\,.$$

Insistons sur le fait que l'inégalité (9) est une inégalité stricte puisqu'en effet, nous souvenant de la définition de ω_0, on voit que l'égalité ne pourrait avoir lieu que si $\omega(k)\Phi(k) = K\Phi(k)$ c'est-à-dire si $\Phi^2(k)$ était une fonction δ, ce qui est impossible d'après la forme que nous avons trouvée pour $\Phi(k)$, où $R(k)$ est essentiellement différent d'une fonction δ. L'égalité $N = M^2/3$ entrainant d'autre part la nullité du premier membre de (10), nous trouvons comme condition nécessaire et suffisante pour que $\sigma^2(H, \psi_T) = 0$, l'ensemble des trois conditions suivantes:

$$(11) \qquad \begin{cases} |\psi_T\rangle \text{ est vecteur propre de } H_T\,, \\ N = \dfrac{M^2}{3}\ (*)\,, \\ \Phi(k) = \dfrac{fMR(k)}{N\omega(k) - \lambda}\,. \end{cases}$$

(*) On peut également voir que la condition $N = M^2/3$ est nécessaire en identifiant les deux impressions de $\Phi(k)$ que l'on obtient en faisant le produit scalaire des deux nombres de (6) per $\langle \psi^* a_i^*|$ et par $\langle \psi^* \tau_i|$. Dans le premier cas on obtient $\Phi(k) = MfR(k)/(N\omega(k) - \lambda)$, dans le deuxième cas $\Phi(k) = 3fR(k)/(N\omega(k) - \lambda')$.

Une condition nécessaire et suffisante pour que $N = M^2/3$ est, d'après (8), que

$$a_i |\psi_T\rangle = C\tau_i |\psi_T\rangle \,, \tag{12}$$

C étant une constante. Si l'on cherche alors C de façon que la solution de l'équation (12) soit vecteur propre de H_T, c'est-à-dire que

$$\omega_0 a_i^* \tau_i C |\psi_T\rangle - f_0 \tau_i a_i^* |\psi_T\rangle - f_0 \tau_i^2 C |\psi_T\rangle = E_T |\psi_T\rangle \,,$$

on voit que la valeur de C « la plus simple », c'est-à-dire qui entraîne que $|\psi_T\rangle$ est vecteur propre de H_T sans que l'on ait à imposer des conditions autres que (12) à $|\psi_T\rangle$ est $C = f_0/\omega_0$; alors l'énergie sera $E_T = -3f_0^2/\omega_0$. On voit donc qu'un choix convenable de C permet d'éviter d'imposer à $|\psi_T\rangle$ la condition d'être vecteur propre de H_T. On peut justifier le choix que nous faisons de C d'une façon plus satisfaisante: la valeur de $\langle \psi_T^* H \psi_T \rangle$ correspondant à $|\psi_T\rangle$ solution de l'équation (12) est $\langle \psi_T^* H \psi_T \rangle = 3\omega_0 C^2 - 6 f_0 C$ qui est minimum pour $C = f_0/\omega_0$. Ainsi l'état *fondamental* de H correspondra a cette valeur particulière de C. Il est intéressant de remarquer que ce n'est que pour le choix de cette constante C que nous faisons l'hypothèse que nous cherchons l'état fondamental du nucélon et non pas un état propre quelconque de H, le reste des calculs est valable également pour un état excité du nucléon.

4. – Etude du système d'équations $a |\psi_T\rangle = (f_0/\omega_0)\tau_i |\psi_T\rangle$ (12').

Nous pouvons remarquer à *priori* que ce système d'équations est incompatible, ce qui traduit le fait que l'approximation de couplage intermédiaire ne donne pas exactement, avec ce couplage, le vecteur d'état fondamental du nucléon. Le système (12') s'écrit, en effet, en introduisant les variables $x_i = (a_i^* + a_i)(1/2^{\frac{1}{2}})$ et en cherchant la fonction inconnue $F(\boldsymbol{x})$ qui remplace alors $|\psi_T\rangle$

$$\frac{\partial F(\boldsymbol{x})}{\partial x_i} = \frac{1}{2^{\frac{1}{2}}} \left(\frac{f_0}{\omega_0} \tau_i - x_i \right) F(\boldsymbol{x}) \,.$$

Une condition nécessaire et suffisante de compatibilité de ce système est

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i} \,,$$

qui équivaut à $\tau_i \tau_j = \tau_j \tau_i$ (dans le cas du couplage scalaire neutre le système correspondant est bien compatible et l'on sait que dans ce cas l'approximation de Tomonaga donne exactement l'état fondamental du nucléon). En mul-

tipliant les deux membres du système (12′) par a_i^*, puis par τ_i on obtient les deux équations.

$$(12'') \qquad a_i^* a_i |\psi_T\rangle = \frac{f_0}{\omega_0} \tau_i a_i^* |\psi_T\rangle \quad (*) ,$$

$$(12''') \qquad \tau_i a_i |\psi_T\rangle = 3 \frac{f_0}{\omega_0} |\psi_T\rangle ,$$

et il est facile de vérifier qu'un vecteur d'état $|\psi_T\rangle$, solution de ces deux équations, sera encore vecteur propre de H_T ainsi que de H. Ceci va nous permettre de donner un sens quantitatif à l'incompatibilité que nous avons signalée: il nous suffira de résoudre l'équation (12″) et d'étudier avec quelle approximation sa solution satisfait l'équation (12‴).

La solution de l'équation (12″) est

$$(13) \qquad |\psi_T\rangle = A \sum \frac{(f_0/\omega_0)^n}{n!} (\tau_i a_i^*)^n |\psi_0\rangle .$$

A est une constante définie par $\langle \psi_T^* \psi_T \rangle = 1$.

Examinons si ce vecteur d'état satisfait à l'équation (12‴). Pour cela nous avons à chercher la valeur de $\tau_i a_i (\tau_j a_j^*)^n |\psi_0\rangle$. Ce calcul est simplifié par le fait que $(\tau_j a_j^*)^{2n} = (a_j^{*2})^n$.

On vérifie alors par récurrence que

$$u_n = \tau_i a_i (\tau_j a_j^*)^{2n} |\psi_0\rangle \quad = 2n (\tau_j a_j^*)^{2n-1} |\psi_0\rangle ,$$

et

$$v_n = \tau_i a_i (\tau_j a_j^*)^{2n+1} |\psi_0\rangle = (2n+3)(\tau_j a_j^*)^{2n} |\psi_0\rangle .$$

Ainsi

$$\tau_i a_i A \sum \frac{(f_0/\omega_0)^n}{n!} (\tau_j a_j^*)^n |\psi_0\rangle = \frac{f_0}{\omega_0} \left(|\psi_T\rangle + 2 |\psi_T'\rangle \right) ,$$

où

$$(14) \qquad |\psi_T'\rangle = A \sum \frac{(f_0/\omega_0)^{2n}}{(2n+1)!} (\tau_j a_j^*)^{2n} |\psi_0\rangle .$$

Etudions ce que devient le second membre de cette relation lorsque $f + (1/f) \to \infty$.

Lorsque $f \to 0$, en nous bornant aux deux premiers termes de la série donnant $|\psi_T\rangle$ (approximation à un méson), nous avons:

$$\tau_i a_i \left(A + A \frac{f_0}{\omega_0} \tau_j a_j^* \right) |\psi_0\rangle = 3A \frac{f_0}{\omega_0} |\psi_0\rangle ,$$

(*) Ceci entraîne $\lambda = 0$.

c'est-à-dire qu'à l'approximation à un seul méson l'équation (12‴) est satisfaite. A cette approximation, $|\psi_T\rangle$ donné par (13) sera vecteur propre de H_T et de H, la valeur de l'énergie est alors $E = -3(f_0^2/\omega_0)$.

Pour une valeur finie de f et pour $f \to \infty$ (dans ce cas $f_0/\omega_0 \to \infty$) nous comparons le vecteur d'état $|\psi_T\rangle$ que nous avons trouvé, à l'état fondamental de H_T, en formant

$$\sigma^2(H_T, \psi_T)\,(*) \qquad \text{et} \qquad \operatorname{tg}^2\gamma = \frac{\sigma^2(H_T\psi_T)}{\langle\psi_T^* H_T\psi_T\rangle^2}\,.$$

En utilisant les relations (12″) et (14) on voit que

$$\sigma^2(H_T, \psi_T) = 4\,\frac{f_0^4}{\omega_0^2}\left(\langle\psi_T'^*\psi_T'\rangle - \langle\psi_T^*\psi_T'\rangle^2\right),$$

et

$$\langle\psi_T^* H_T\psi_T\rangle = -\,\frac{f_0^2}{\omega_0}\left(1 + 2\langle\psi_T^*\psi_T'\rangle\right).$$

Les expressions de u_n et v_n permettent de calculer la constante de normalisation A, $\langle\psi_T^*\psi_T'\rangle$ et $\langle\psi_T'^*\psi_T'\rangle$.

On a en effet:

$$\langle\psi_0^*(a_i^2)^n\,(a_j^{*2})^n\,\psi_0\rangle = 2n(2n+1)\langle\psi_0^*(a_i^2)^{n-1}(a_j^{*2})^{n-1}\psi_0\rangle\,,$$

d'où

$$\langle\psi_0^*(a_i^2)^n\,(a_i^{*2})^n\,\psi_0\rangle = (2n+1)!\,,$$

et l'expression que nous avons trouvé pour

$$v_n = \tau_i a_i(\tau_j a_j^*)^{2n+1}|\psi_0\rangle = (2n+3)(a_j^{*2})^n|\psi_0\rangle\,,$$

donne alors

$$\langle\psi_0^*(\tau_i a_i)^{2n+1}(\tau_j a_j^*)^{2n+1}|\psi_0\rangle = (2n+3)(2n+1)!\,,$$

(*) C'est à l'état fondamental de l'Hamiltonien H_T qui correspond à la fonction de poids $\Phi(k) = f\omega_0 R(k)/f_0\omega(k)$ que nous comparons $|\psi_T\rangle$. Il faudrait pour comparer $|\psi_T\rangle$ au vecteur d'état donné par la méthode de Tomonaga, considérer l'Hamiltonien qui correspond à la fonction de poids $\Phi(k) = fMR(k)/(N\omega(k) - \lambda)$, où λ a la valeur (en général non nulle) qui correspond au minimum de la fonctionnelle $E_T(\Phi)$. Mais on sait que lorsque $f \to \infty$, $\lambda \to 0$ et que pour une valeur utile de f, d'après une remarque de Lee (citée dans un article de Stroffolini [4]), on ne change pas sensiblement la valeur de E_T en remplaçant λ par 0.

[4] R. Stroffolini: *Phys. Rev.*, **104**, 1146 (1956).

d'où en posant $(f_0/\omega_0)^2 = Z$

$$\frac{1}{A^2} = \sum \left[\frac{(f_0/\omega_0)^{4n}}{2n!}(2n+1) + \frac{(f_0/\omega_0)^{4n+2}}{(2n+1)!}(2n+3) \right] = (Z+1)\cosh Z + (Z+2)\sinh Z,$$

$$\langle \psi'^* \psi' \rangle = A^2 \sum \frac{(f_0/\omega_0)^{4n}}{(2n+1)!} = \frac{\sinh Z}{Z(Z+1)\cosh Z + Z(Z+2)\sinh Z},$$

$$\langle \psi^* \psi' \rangle = A^2 \sum \frac{(f_0/\omega_0)^{4n}}{2n!} = \frac{\cosh Z}{(Z+1)\cosh Z + (Z+2)\sinh Z}.$$

$$\langle \psi_T^* H_T \psi_T \rangle = -\omega_0 z \left(1 + \frac{2\cosh Z}{(z+1)\cosh Z + (Z+2)\sinh Z} \right),$$

Ainsi

$$\sigma^2(H_T, \psi_T) = 4 \left(\frac{\sinh Z}{Z(Z+1)\cosh Z + Z(Z+2)\sinh Z} - \frac{\cosh^2 Z}{[(Z+1)\cosh Z + (Z+2)\sinh Z]^2} \right) Z^2.$$

On voit sur ces expressions que

$$\operatorname{tg}^2 \gamma = \frac{\sigma^2(H_T, \psi_T)}{\langle \psi_T^* H_T \psi_T \rangle^2},$$

tend vers 0 pour $f + (1/f) \to \infty$ comme indiqué sur les courbes ci-après.

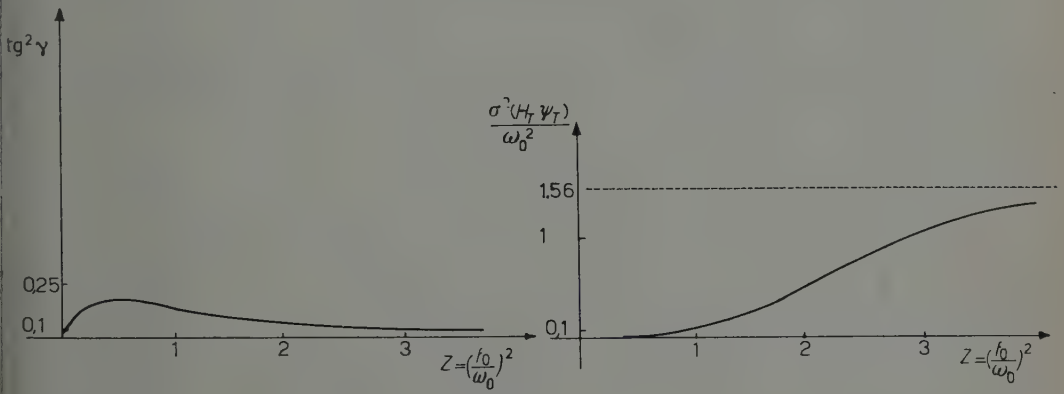

Fig. 1.

Nous voyons en particulier que lorsque $f \to \infty$ le vecteur d'état $|\psi_T\rangle$ donné par la formule (13) fait un angle γ avec $|H_T\psi_T\rangle$ qui tend vers zéro comme $1/f^2$ alors que d'après (5') il fait un angle γ' avec son transformé $|H\psi_T\rangle$ qui tend vers zéro comme $1/f$. Ceci veut dire que pour $f \to \infty$ le

vecteur d'état $|\psi_T\rangle$ que nous avons trouvé devient vecteur d'état fondamental de H, il en est donc de même du vecteur d'état donné par l'approximation de Tomonaga, mais le vecteur d'état $|\psi_T\rangle$ que nous indiquons tend plus rapidement (avec un ordre en $1/f$ plus élevé) vers le vecteur d'état de l'approximation de Tomonaga, que celui-ci ne tend vers le vecteur d'état fondamental vrai du nucléon. Une conséquence pratique de ceci est que dès que f est suffisamment grand (l'intensité de la valeur de f nécessaire dépendant du rapport de f à f_0/ω_0, c'est-à-dire du cut-off, et de la précision demandée), l'approximation simplifiée de couplage intermédiaire que nous indiquons donne une précision comparable à celle de l'approximation de couplage intermédiaire véritable.

5. – Conclusion.

L'étude de la validité de l'approximation de couplage intermédiaire nous conduit à proposer une forme analytique approchée du vecteur d'état fondamental du nucléon, qui devient colinéaire au vecteur d'état de l'approximation de couplage intermédiaire lorsque $f+(1/f) \to \infty$, l'angle de ces deux vecteurs d'état tendant vers zéro comme f lorsque $f \to 0$ et comme $1/f^2$ lorsque $f \to \infty$. Les mêmes calculs s'appliquent au couplage pseudovectoriel (5) symétrique et donnent, dans ce cas, comme vecteur d'état fondamental approché du nucléon

$$|\psi_T\rangle = \beta \sum \frac{(f_0/\omega_0)^n}{n!} (\sigma_i \tau_\alpha a^*_{i\alpha})^n |\psi_0\rangle ,$$

(les opérateurs $a^*_{i\alpha}$ analogues de a^*_i sont définis dans la référence (5)) forme qui avait déjà été utilisée par M. Lévy [6].

([5]) M. H. Friedman, T. D. Lee et R. Christian: *Phys. Rev.*, **100**, 1494 (1955).
([6]) M. Lévy: *Nuovo Cimento*, **8**, 92 (1958).

RIASSUNTO (*)

Si discute in modo critico l'approssimazione dell'accoppiamento per tutti i valori della costante di accoppiamento. Si fornisce una approssimazione semplificata che possiede proprietà simili e si risolvono in modo esplicito le equazioni corrispondenti.

(*) *Traduzione a cura della Redazione.*

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K^--Nucleon Scattering Lengths and the K^--d Scattering Reactions (*).

T. B. Day, G. A. Snow and J. Sucher

Department of Physics, University of Maryland - College Park, Md.

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Summary. — The K^--d elastic and total cross sections are calculated, using a model which includes multiple scattering effects. The scattering lengths which Dalitz and Tuan find as fits to the K^--p data are used in an attempt to distinguish between the four possibilities. The impulse approximation fails to give sensible results in this problem.

1. – Introduction.

Recently, Dalitz and Tuan [1] have analyzed the K^--nucleon scattering data at low energies. Following a suggestion of Jackson *et al.* [2], they represent these data [3] by two complex scattering lengths A_0 and A_1, one each for the $I=0$ and $I=1$ isotopic spin scattering channels. These amplitudes are related to the complex phase shifts δ_I by

$$k \cot \delta_I = 1/A_I(k) \tag{1}$$

and the momentum dependence of the A_I is neglected.

(*) This research was supported in part by the U. S. Atomic Energy Commission and by the U. S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

(1) R. H. Dalitz and S. F. Tuan: *Phys. Rev. Lett.*, **2**, 425 (1959).

(2) J. C. Jackson, D. G. Ravenhall and H. W. Wyld jr.: *Nuovo Cimento*, **9**, 834 (1958).

(3) W. Alvarez, H. Bradner, P. Falk-Vairant, J. D. Gow, A. H. Rosenfeld, F. T. Solmitz, R. D. Tripp and K. M. Watson: *Phys. Rev. Lett.*, **2**, 312 (1959); P. P. Nordin, A. H. Rosenfeld, F. T. Solmitz, R. D. Tripp and K. M. Watson: *Bull. Am. Phys. Soc.*, **4**, 288 (1959).

Then the data admit of four solutions (in units of fermis):

$$(2)\qquad \begin{cases} A_0 = (0.20 + 0.78i)\,, & A_1 = (1.62 + 0.39i) \quad (a+) \\ A_0 = (1.88 + 0.82i)\,, & A_1 = (0.40 + 0.41i) \quad (b+) \end{cases}$$

and the solutions $(a-)$ and $(b-)$ obtained from $(a+)$ and $(b+)$ by reversing the sign of the real parts of both A_0 and A_1.

In an attempt to distinguish between these four solutions, DALITZ and TUAN [4] have calculated the K^--p elastic scattering total cross-sections for rather low values of the K^- laboratory momentum, neglecting Coulomb effects. Below the laboratory threshold for $\overline{K}^0$ production (90 MeV/c) the four sets show their greatest differences. However, in this momentum range it is very difficult to get significant data, and the errors in existing data are larger than the difference between the solutions. Above the $\overline{K}^0$ production threshold, the plus and minus solutions merge, and the difference between $(a\pm)$ and $(b\pm)$ becomes small. JACKSON and WYLD [5] have calculated the K^--p elastic scattering cross-sections including the Coulomb effects, again in the momentum region around the $\overline{K}^0$ production threshold. They find that the solutions $(a-)$ and $(b-)$ more nearly follow the emulsion data [6], owing to the destructive interference with the Coulomb scattering. As both groups of authors point out, small-angle K^--p elastic scattering will determine the sign of the real parts of A_0 and A_1 from the nucleon-Coulomb interference.

However, as is also pointed out by DALITZ and TUAN [4], and calculated by HILL [7] for the solutions $(b\pm)$, a quite large difference between the solutions is predicted for K^--n elastic scattering at higher K^- momenta (although still small enough to be predominantly s-wave interaction). Since such a region of momenta around (200÷250) MeV/c is more amenable to experiment, and since the simplest system containing neutrons is the deuteron, we have felt it worth-while to see what might be learned from a study of the K^--d elastic scattering and total cross-sections.

Within the framework of the impulse approximation, these cross-sections may be simply related to the measured K^--nucleon cross-sections. However, as pointed out by DALITZ and TUAN [4], since the cross-sections are so large multiple scattering effects are expected to be large. We therefore attempt to

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[4] R. H. DALITZ and S. F. TUAN: *Ann. Phys.* (to be published).

[5] J. C. JACKSON and H. W. WYLD jr.: *Phys. Rev. Lett.*, **2**, 355 (1959).

[6] R. ASCOLI, R. D. HILL and T. S. YOON: *Nuovo Cimento*, **9**, 313 (1958); W. ALLES, N. N. BISWAS, M. CECCARELLI, R. GESSAROLI, G. QUARENI, M. GÖING, K. GOTTSTEIN, W. PÜSCHEL, J. TIETGE, G. T. ZORN, J. CRUSSARD, J. HENNESSY, G. DASCOLA and S. MORA: *Nuovo Cimento*, **11**, 771 (1959).

[7] R. D. HILL: Technical Report no. 7 (University of Illinois, 1959).

incorporate some of these effects by use of a simple model which still allows the use of the two-body data. In particular, we have used the point-scatterer model used by BRUECKNER ([8]) in studying the analogous π^--d reactions.

The calculations are presented in Section **2**, the results are give in Section **3**, and discussed in Section **4**.

2. – Calculations.

BRUECKNER has given a formula for the scattering of a particle from two identical infinitely heavy, s-wave « point » scatterers (interaction range is short compared to the separation of the scatterers) ([8]). The generalization to the case of non-identical scatterers is trivial, and gives for the scattering amplitude from a deuteron (where θ is the angle of scattering of the meson in the K⁻-d center of mass system)

$$f(\theta) = \langle f(\theta; R)\rangle = \int |u(\boldsymbol{R})|^2 f(\theta; R)\, \mathrm{d}\boldsymbol{R}\,, \tag{3}$$

where $u(R)$ is the deuteron (Hulthén) wave function. $f(\theta; R)$ is given by

$$f(\theta; R) = N/D\,, \tag{4}$$

where

$$N = (\eta_n + \eta_p)\frac{\sin(qR/2)}{(qR/2)} + 2\eta_n\eta_p\frac{\exp[ikR]}{R}\frac{\sin QR}{QR}\,, \tag{5}$$

and

$$D = 1 - \eta_n\eta_p\frac{\exp[2ikR]}{R^2}\,, \tag{6}$$

with

$$q = 2k\sin\theta/2\,, \qquad Q = k\cos\theta/2\,. \tag{7}$$

Here, k is the final K⁻ momentum in the K⁻-d center of mass system. In equations (5) and (6), η_n and η_p are the K⁻-n and K⁻-p scattering amplitudes (in the deuteron) and are related to the free K⁻-nucleon amplitudes (f_n and f_p) by

$$\eta_n = (\mu_{kd}/\mu_{kn})f_n\,, \qquad \eta_p = (\mu_{kd}/\mu_{kp})f_p\,, \tag{8}$$

([8]) K. A. BRUECKNER: *Phys. Rev.*, **89**, 128 (1953); **90**, 715 (1953).

where the μ's are appropriate reduced masses. f_n and f_p are given in terms of the scattering lengths A_0, A_1 by

$$(9) \qquad \begin{cases} f_p = \frac{1}{2}\left(\frac{A_1}{1 - ik'_0 A_1} + \frac{A_0}{1 - ik'_0 A_0}\right), \\ f_n = \frac{A_0}{1 - ik'_0 A_0}. \end{cases}$$

k'_0 is the K^- momentum in the K^--nucleon center of mass system [9], (k_L is the K^- momentum in the laboratory system)

$$(10) \qquad k'_0 = \frac{m_p}{m_k + m_p} k_L,$$

compared with

$$(11) \qquad k = \frac{m_d}{m_k + m_d} k_L,$$

for the k in equation (7). After evaluation of $f(\theta)$, from eq. (3), the differential elastic scattering cross-section, as well as the total cross-section are simply calculated:

$$(12) \qquad \frac{d\sigma_{elastic}}{d\Omega} = |f(\theta)|^2,$$

and

$$(13) \qquad \sigma_{total} = (4\pi/k) \operatorname{Im} f(0).$$

The evaluation of the integral in equation (3) was done using the following successively more accurate approximations:

i) Impulse approximation;

$$(14) \qquad f(\theta) \sim f_{imp.}(\theta) \equiv (\eta_n + \eta_p) \left\langle \frac{\sin(qR/2)}{(qR/2)} \right\rangle.$$

ii) Double scattering approximation;

$$(15) \qquad f(\theta) \sim \langle N \rangle \equiv f_{imp.}(\theta) + 2\eta_n \eta_p \left\langle \frac{\exp[ikR]}{R} \frac{\sin QR}{QR} \right\rangle.$$

[9] The spread in k'_0 for fixed k_L, due to the momentum spread of the deuteron, is ignored.

iii) Multiple scattering to all orders, « approximate »;

$$f(\theta) \sim \langle N \rangle / \langle D \rangle \,. \tag{16}$$

iv) Multiple scattering to all orders, « exact »

$$f(\theta) = \langle N/D \rangle \,. \tag{17}$$

For the first three approximations, the average in equation (3) can be done analytically, and some pertinent formulae are listed in Appendix. In the « exact » evaluation, iv) the average must be performed numerically.

3. – Results.

The evaluation of $f(\theta)$ for the first three approximations was performed for all four solutions at three values of k_L ($=200$, 260, 327 MeV/c), each for three angles θ ($\theta = 0°$, 90°, 180°). θ is the meson scattering angle in K⁻-d

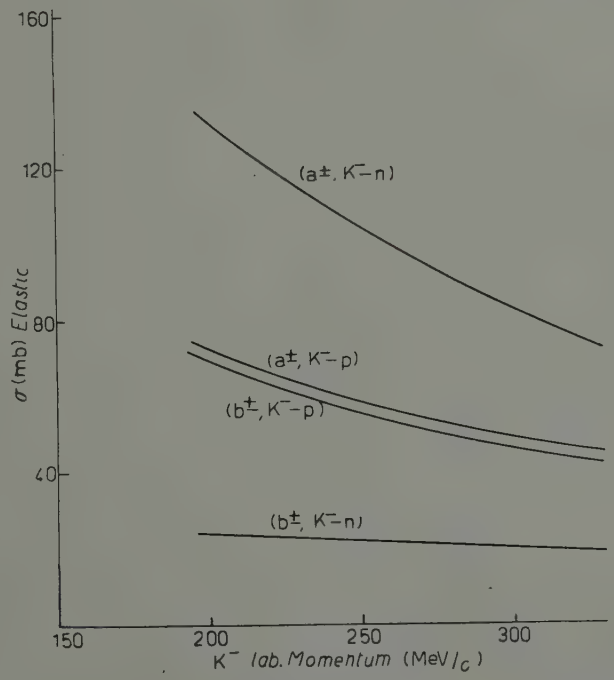

Fig. 1. – K⁻-n and K⁻-p elastic scattering cross sections (mb) plotted against momentum of the K⁻ in the laboratory system (MeV/c). Curves are drawn for the Dalitz and Tuan solutions ($a\pm$) and ($b\pm$); see ref. ([1]).

center of mass system. The numerical integration for step (iv) was performed only for the Dalitz solutions $a+$ and $a-$ at $k_L = 200$ MeV/c. Values for σ_{elastic} were obtained from numerical integration of the $d\sigma_{\text{elastic}}/d\Omega$ curves.

TABLE I. – K^--d *cross sections*: $k_L = 200$ MeV/c.

		Impulse	Double	Multiple (approx.)	Multiple (exact)	
$a+$	$(d\sigma/d\Omega)$ 0°	47.3	45.5	18.1	29.1	mb/sr
	$(d\sigma/d\Omega)$ 90°	28.4	14.6	5.82	5.56	
	$(d\sigma/d\Omega)$ 180°	20.0	7.80	3.10	1.32	
	σ_{elastic}	390	233	94.0	116	mb
	σ_{total}	290	332	214	246	
$a-$	$(d\sigma/d\Omega)$ 0°	47.3	20.7	27.3	34.9	mb/sr
	$(d\sigma/d\Omega)$ 90°	28.4	1.86	2.00	4.41	
	$(d\sigma/d\Omega)$ 180°	20.0	.033	.035	1.64	
	σ_{elastic}	390	63.5	68.4	119	mb
	σ_{total}	290	178	234	267	
$b+$	$(d\sigma/d\Omega)$ 0°	21.9	20.2	13.0		mb/sr
	$(d\sigma/d\Omega)$ 90°	13.0	4.69	3.02		
	$(d\sigma/d\Omega)$ 180°	9.3	2.77	1.79		
	σ_{elastic}	188	91.1	58.4		mb
	$\sigma_{\text{tot 1}}$	200	215	175		
$b-$	$(d\delta/d\Omega)$ 0°	21.9	12.8	14.3		mb/sr
	$(d\delta/d\Omega)$ 90°	13.0	2.39	2.68		
	$(d\delta/d\Omega)$ 180°	9.3	.54	.60		
	σ_{elastic}	188	51.1	54.5		mb
	σ_{total}	200	148	176		

The results of the calculations are presented in Tables I, II and III. For purposes of orientation, the total elastic scattering cross-sections for K^--nucleon scattering are given in Fig. 1. In the first step of the calculation (the impulse approximation) the Coulomb interference term was included, and the re-

sulting curves for $d\sigma_{elastic}/d\Omega$ showed very nicely the separation of the plus and minus solutions similar to those obtained by JACKSON and WYLD [5]. Since, however, the later steps of the calculation showed that the impulse approximation is essentially meaningless here. (*E.g.* use of the optical theorem

TABLE II. – K⁻-d *cross sections*: $k_L = 260$ MeV/c.

		Impulse	Double	Multiple (approx.)	
$a+$	$(d\sigma/d\Omega)$ 0°	44.0	29.2	15.8	mb/sr
	$(d\sigma/d\Omega)$ 90°	19.0	5.58	3.02	mb/sr
	$(d\sigma/d\Omega)$ 180°	11.0	1.30	0.70	mb/sr
	$\sigma_{elastic}$	245	111	61.4	mb
	σ_{total}	210	206	153	mb
$a-$	$(d\sigma/d\Omega)$ 0°	44.0	19.7	21.2	mb/sr
	$(d\sigma/d\Omega)$ 90°	19.0	2.16	2.32	mb/sr
	$(d\sigma/d\Omega)$ 180°	11.0	0.03	0.03	mb/sr
	$\sigma_{elastic}$	245	62.1	66.0	mb
	σ_{total}	210	150	177	mb
$b+$	$(d\sigma/d\Omega)$ 0°	19.5	15.1	11.1	mb/sr
	$(d\sigma/d\Omega)$ 90°	7.90	2.94	2.17	mb/sr
	$(d\varrho/d\Omega)$ 180°	5.20	0.73	0.54	mb/sr
	$\sigma_{elastic}$	115	53.0	43.7	mb
	σ_{total}	145	144	124	mb
$b-$	$(d\sigma/d\Omega)$ 0°	19.5	12.0	12.9	mb/sr
	$(d\sigma/d\Omega)$ 90°	7.90	1.73	1.86	mb/sr
	$(d\sigma/d\Omega)$ 180°	5.20	0.20	0.22	mb/sr
	$\sigma_{elastic}$	115	38.2	43.9	mb
	σ_{total}	145	118	132	mb

gives absurd results in certain cases; all attempts to incorporate the Coulomb effects were dropped, and the results given in the tables do not include them. Thus, no cut-off angle is necessary, and the total elastic cross-sections include contributions from all angles. The contribution to $\sigma_{elastic}$ from angles with $\cos\theta \geqslant 0.8$ is about $(20 \div 30)\%$ when multiple scattering is included.)

TABLE III. – K^--d *cross sections*: $k_L = 327$ MeV/c.

		Impulse	Double	Multiple (approx.)	
$a+$	$(d\sigma/d\Omega)$ 0°	31.6	22.6	15.9	mb/sr
	$(d\sigma/d\Omega)$ 90°	10.2	2.40	1.69	mb/sr
	$(d\sigma/d\Omega)$ 180°	6.2	0.30	0.21	mb/sr
	$\sigma_{elastic}$	155	65.3	48.1	mb
	σ_{total}	155	144	118	mb
$a-$	$(d\sigma/d\Omega)$ 0°	31.6	17.6	17.4	mb/sr
	$(d\sigma/d\Omega)$ 90°	10.2	1.61	1.59	mb/sr
	$(d\sigma/d\Omega)$ 180°	6.2	0.15	0.15	mb/sr
	$\sigma_{elastic}$	155	49.4	49.8	mb
	σ_{total}	155	121	132	mb
$b+$	$(d\sigma/d\Omega)$ 0°	15.9	12.3	10.3	mb/sr
	$(d\sigma/d\Omega)$ 90°	5.0	1.47	1.24	mb/sr
	$(d\sigma/d\Omega)$ 180°	3.1	0.16	0.13	mb/sr
	$\sigma_{elastic}$	77.4	38.2	32.4	mb
	σ_{total}	110	105	95.0	mb
$b-$	$(d\sigma/d\Omega)$ 0°	15.9	10.5	10.7	mb/sr
	$(d\sigma/d\Omega)$ 90°	5.0	1.12	1.14	mb/sr
	$(d\sigma/d\Omega)$ 180°	3.1	0.11	0.11	mb/sr
	$\sigma_{elastic}$	77.4	32.9	33.4	mb
	σ_{total}	110	93.0	99.5	mb

4. – Discussion and conclusions.

The information contained in Tables I, II and III, will now be discussed in some detail. The first, most obvious point, is that the impulse approximation is very bad. Note, in particular, that the results for solution $a\pm$ (identical in the impulse approximation, neglecting Coulomb effects), are such that $\sigma_{elastic} \geqslant \sigma_{total}$ for the three momenta used. Even for solutions $b\pm$, where $\sigma_{elastic} < \sigma_{total}$ the difference between σ_{total} and $\sigma_{elastic}$ is implausibly small.

However, as soon as the next simplest approximation is made, the « double scattering » approximation, this difficulty is removed. (Note also, in this ap-

proximation, that the plus and minus solutions now split.) Here, too, another feature of the inclusion of multiple scattering effects is revealed, namely the depression of the backward elastic scattering compared to that given by the impulse approximation. (Compare this feature with a similar one in the case of π-d scattering considered by BRUECKNER ([8]).)

In taking the next step of the calculation, going from double scattering to multiple scattering « approximate », the tables show that there is still a significant change. This is in contrast to the pion case ([10]). It is seen that these changes are smaller for the $b+$ and $b-$ solutions, which is consistent with the smaller scattering amplitudes associated with these cases.

The tables show that in going from the impulse approximation to any one of the more accurate approximations, all differential and total elastic cross-sections decrease.

The splitting between plus and minus solutions, (particularly in the one case calculated for the « multiple exact ») is relatively small, while that between the a solutions and the b solutions is more pronounced, both for σ_{elastic} and for σ_{total}.

While Table I shows that for the a solutions at 200 MeV/c, the « multiple approximate » calculation is significantly different from the « multiple exact » it is felt that this would not be true for the b solutions at this momentum (particularly for $b-$). Therefore, the « multiple approximate » results should be representative of the results of a still more exact calculation.

If we now concentrate on the $k_L = 200$ MeV/c case, and compare the « multiple exact » for $a\pm$ with the « multiple approximate » for $b\pm$ (for the reasons given above) we see that, at least within the framework of the model used, the difference between solutions a and b is considerable, so that experiment should be able to distinguish between them. For example, both σ_{elastic} and σ_{total} change by 50 mb in going from a to b. However, it should be emphasized that an unambiguous measurement of σ_{elastic} is quite difficult, since spurious events due to inelastic « break-up » of the deuteron are difficult to exclude. Since an estimate of this inelastic « break-up » cross-section has not been made as yet ([11]), it is difficult to compare our theoretical results for the a and b solutions with experimental data that do not distinguish between elastic and inelastic K^- scattering. A measurement of σ_{total}, however, would not be subject to such a difficulty.

Finally, it should be emphasized that the usefulness of such calculations is limited by the sensitivity of the results to the model assumed for the de-

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([10]) R. M. ROCKMORE: *Phys. Rev.*, **105**, 256 (1957).

([11]) An estimate of the « break-up » cross section is easily made, in the impulse approximation, which from the above would clearly not be very reliable. A calculation of this cross section including the multiple scattering effects is in progress.

scription of the multiple scattering ([12]). If it can be shown that at least the trends observed above remain unchanged if widely different models are assumed, and this is not at all clear *a priori* ([13]) because of the largeness of the multiple scattering effects, the observation of K^--d scattering reactions offers some hope for distinguishing between the Dalitz solutions ([14]).

* * *

We would like to thank Mr. A. K. BHATIA for his assistance with the numerical work involved in the « multiple exact » calculations. We are also indebted to Dr. D. MILLER for a discussion of several aspects of the experiment on (K^-, d) interactions in progress.

APPENDIX

If, for the deuteron wave function, we take

$$u(\boldsymbol{R}) = N_0 \left(\frac{\exp[-\alpha k] - \exp[-\beta R]}{R} \right), \tag{A.1}$$

where

$$\begin{cases} N_0^2 = (\alpha/2\pi)(1-\alpha\varrho)^{-1} = (\alpha/2\pi)(0.606)^{-1}, \\ \beta = 6.2\alpha, \\ \alpha = 45.6 \text{ MeV/c} = 0.231 \text{ fermi}^{-1}, \end{cases} \tag{A.2}$$

([12]) Even in the impulse approximation, there remain the difficulties inherent in continuing the two-body scattering matrix off the energy shell. Compare with T. A. GREEN: *Phys. Rev.*, **90**, 161 (1953). These difficulties are accentuated in our case, because of the larger mass of the K-meson; however, the K^--nucleon scattering cross sections are rather slowly varying with energy, this may partly compensate for the errors introduced by using only « on the energy shell » two-body scattering matrices.

([13]) S. D. DRELL and L. VERLET: *Phys. Rev.*, **99**, 849 (1955).

([14]) A rough calculation of the multiple scattering « exact » for a different model, namely, one in which the meson propagator is $(i \sin kR)/R$ (compared to $\exp[ikR/R]$ in the body of this paper) has been made. This propagator corresponds, in a sense, to no scattering « off the energy shell » (see ref. ([13])). The results for the $a+$ solution $k_L = 200$ MeV/c are $d\sigma_{\text{elastic}}/d\Omega = 34.9$ at 0°, 8.29 at 90° and 2.66 at 180°, in mb/sr; $\sigma_{\text{elastic}} = 156$ mb; $\sigma_{\text{total}} = 250$ mb.

then the evaluation for the average in eq. (3) gives the following, (k, q and Q are defined in eq. (7)):

(i) (A.3)
$$f_{\text{imp.}}(\theta) = (\eta_n + \eta_p)(4\alpha/q)(1 - \alpha\varrho)^{-1} \cdot \cdot \left[\operatorname{tg}^{-1}\frac{q}{4\alpha} + \operatorname{tg}^{-1}\frac{q}{4\beta} - 2\operatorname{tg}^{-1}\frac{q}{2(\alpha+\beta)}\right],$$

(ii) (A.4)
$$f_{\text{double}}(\theta) = f_{\text{imp.}}(\theta) + 2\eta_n\eta_p g(\theta),$$

with

(A.5)
$$g(\theta) = \operatorname{Re} g(\theta) + i \operatorname{Im} g(\theta),$$

and

(A.6)
$$\operatorname{Re} g(\theta) = \frac{2\alpha}{Q(1-\alpha\varrho)}\left\{-\alpha \operatorname{tg}^{-1}\frac{Q+k}{2\alpha} - \beta \operatorname{tg}^{-1}\frac{Q+k}{2\beta} + \right.$$
$$+ (\alpha+\beta)\operatorname{tg}^{-1}\frac{Q+k}{\alpha+\beta} + \frac{Q+k}{4}\ln\frac{[(\alpha+\beta)^2 + (Q+k)^2]^2}{[4\alpha^2 + (Q+k)^2][4\beta^2 + (Q+k)^2]} +$$
$$\left. + (\text{similar terms with } k \to -k)\right\},$$

(A.7)
$$\operatorname{Im} g(\theta) = \frac{2\alpha}{Q(1-\alpha\varrho)}\left\{\frac{-\alpha}{2}\ln\frac{4\alpha^2 + (Q+k)^2}{4\alpha^2 + (Q-k)^2} + k\left[\frac{1}{2}\operatorname{tg}^{-1}\frac{4\alpha Q}{4\alpha^2 - Q^2 + k^2} + \right.\right.$$
$$+ \varepsilon(2\alpha, Q, k)(\pi/2) + Q\left[\frac{1}{2}\operatorname{tg}^{-1}\frac{4\alpha k}{4\alpha^2 - k^2 + Q^2} + \varepsilon(2\alpha, k, Q)(\pi/2)\right] +$$
$$\left.\left. + (\text{similar terms with } \alpha \to \beta) - 2\left(\text{similar terms with } \alpha \to \frac{\alpha+\beta}{2}\right)\right\}.$$

In eq. (A.7),

(A.8)
$$\begin{cases} \varepsilon(a,b,c) = 1, & a^2 - b^2 + c^2 < 0, \\ = 0, & a^2 - b^2 + c^2 \geqslant 0. \end{cases}$$

(iii) (A.9)
$$f_{\text{mult. app.}}(\theta) = f_{\text{double.}}(\theta)/\langle D\rangle,$$

and

(A.10)
$$\langle D\rangle = 1 - \eta_n\eta_p\langle \exp[2ikR]/R^2\rangle,$$

(A.11)
$$\langle \exp[2ikA]/R^2\rangle = \operatorname{Re}\langle\,\rangle + i \operatorname{Im}\langle\,\rangle,$$

with

$$\text{(A.12)} \qquad \mathrm{Re}\langle\rangle = \frac{2\alpha}{1-\alpha\varrho}\left\{-2k\,\mathrm{tg}^{-1}\frac{k(\beta-\alpha)}{\alpha(\alpha+\beta)+2k^2} - 2k\,\mathrm{tg}^{-1}\frac{k(\alpha-\beta)}{\beta(\alpha+\beta)+2k^2} - \right.$$
$$\left. - \alpha\ln\frac{(\alpha+\beta)^2+4k^2}{4\alpha^2+4k^2} - \beta\ln\frac{(\alpha+\beta)^2+4k^2}{4\beta^2+4k^2}\right\},$$

$$\text{(A.13)} \qquad \mathrm{Im}\langle\rangle = \frac{2\alpha}{1-\alpha\varrho}\left\{-2\alpha\,\mathrm{tg}^{-1}\frac{k}{\alpha} - 2\beta\,\mathrm{tg}^{-1}\frac{k}{\beta} + \right.$$
$$\left. + 2(\alpha+\beta)\,\mathrm{tg}^{-1}\frac{2k}{\alpha+\beta} + k\ln\frac{[(\alpha+\beta)^2+4k^2]^2}{[4\alpha^2+4k^2][4\beta^2+4k^2]}\right\}.$$

Note added in proof.

With reference to footnote (11): We have completed a calculation of the cross section (σ_{e+i}) for the sum of elastic, and inelastic « break-up » scattering of K^--mesons from deuterons. An extension of the Brueckner multiple-scattering model described above was used, while in the actual evaluation, a closure approximation was employed for a propagator ($i \sin kR/R$). Details are given in University of Maryland Physics Department Technical Report no. 146. The results at a K^- momentum in the laboratory of 200 MeV/c are: solutions $a_\pm$, $\sigma_{e+i}=163$ mb; solutions $b_\pm$, $\sigma_{e+i}=9.9$ mb.8 Thus, of the Dalitz solutions used in this paper, the new Berkeley data (Dr. D. MILLER: private communication) would favor solutions $b_\pm$.

However, it should be strongly emphasized that the re-analysis of the Berkeley (K^-, p) data presented by Prof. L. ALVAREZ at the High Energy Physics Conference at Kiev, U.S.S.R. in July, 1959 shows that the Dalitz solutions used here are not at all as precisely determined as was thought when this calculation was performed. Rather the errors in the data are such at this time that the above solutions can only be considered as tentative. Thus, the above calculations should not be considered as evidence for the $b_\pm$ solutions unless and until a re-derivation from more extensive data shows that the solutions used above are sufficiently near the true solutions.

RIASSUNTO (*)

Si calcolano le sezioni d'urto K^--d elastiche e totali facendo uso di un modello che includa gli effetti multipli di scattering. Nel tentativo di distinguere fra le quattro possibilità, si usano le lunghezze di scattering che DALITZ e TUAN trovarono essere adattabili ai dati dello scattering K^--p. L'approssimazione dell'impulso è insufficiente a dare in questo problema risultati coerenti.

(*) *Traduzione a cura della Redazione*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Analytic $3d$ Wave-Functions for Atoms of the Iron Group for Small and Large Distances from the Nuclei.

C. Caviglia and I. Fidone (*)

Istituto di Fisica dell'Università - Palermo

(ricevuto il 16 Maggio 1959)

The calculation of the Slater-Condon parameters F_k for the configuration $3d^n$, using Slater type functions, has recently become of interest for elements of the iron group (1). These functions, which are only a rough approximation of the self-consistent wave-functions, give a fairly good agreement between theory and experiment. The object of the present work is to try to get a better agreement in the case of doubly ionized elements of the iron group.

The best wave-functions are those deduced by Hartree's method, but these are known for only a few elements and, moreover, require numerical integration. In order to avoid this integration in the cases in which the wave-functions are known and to deduce the wave-functions for the elements for which self-consistent calculations have not been performed, analytical expressions must be worked out.

Wave-functions of analytical form for the elements of the iron group, based on the preliminary results of Hartree (2) have been obtained by Löwdin and Appel (3), and are expressed as the sum of four exponentials. We have here attempted to obtain an analytical expression containing less than four exponentials and based on more accurate self-consistent calculations. Although three exponentials have been used for the wave-functions in this work, it was found that two exponentials sufficed for a good approximation. Moreover, calculations have been made for the mean value of r^{-3} to test the usefulness of the deduced wave-functions for small values of r (4).

The starting point in our case is the table of numerical values of the $3d$ function of doubly-ionized vanadium obtained by Worsley (5) using the self-consistent

(*) Now at Research Dept., Euratom, Brussels.

(1) D. A. Brown: *Journ. Chem. Phys.*, **28**, 67 (1958).

(2) D. R. Hartree: *The Calculation of Atomic Structures* (New York, N.Y., 1956).

(3) P. Löwdin and K. Appel: *Phys. Rev.*, **403**, 1746 (1956)

(4) I. Fidone: *Nuovo Cimento*, **11**, 736 (1959).

(5) B. H. Worsley: *Proc. Roy. Soc.*, **247**, 390 (1958).

method. These numerical values have been here approximated by an analytical expression of the form

$$f_{3d}(r) = r^3(A \exp[-ar] + B \exp[-br] + C \exp[-cr]),$$

in which the values of the constants were chosen as follows

$$a = 7.50 \qquad b = 3.33 \qquad c = 1.00,$$

$$A = 29.40 \qquad B = 22.91 \qquad C = 0.148.$$

This expression is normalized to unity and agrees satisfactorily with the numerical values in the table as far as $r \simeq 3$ a.u., the maximum difference occurs at $r \simeq 2$ and is about 0.08, our value being lower than that of Worsley. For $r > 3$ the agreement is not so good, our values being in every case greater than those of Worsley. The maximum difference occurs at $r \simeq 4.5$ and is about 0.15.

In the expression for $f_{3d}(r)$ the second term is predominant. The first term has a maximum value of the order of 0.09 at $r \simeq 0.4$ and is therefore important for small values of r. The third term has a maximum value of the order of 0.19 at $r=3$ and is important for large values of r.

To obtain the $3d$ function for other elements of the group the method of « pure scaling » was used ([2]). Let us define a linear factor of scale R for a wave-function of given (nl) and let us introduce a screening number σ by the following expression

$$\sigma = Z - \frac{R_H}{R},$$

where Z is the atomic number of the element in question and R_H is the corresponding R of the hydrogen atom. If the wave-function $f_{3d}(Z_0, r)$ is known for the element with atomic number Z_0, the wave-function $f_{3d}(Z, r)$ for the element of atomic number Z can be obtained from the following

$$(1) \qquad f_{3d}(Z, r) = \left(\frac{Z-\sigma}{Z_0-\sigma}\right)^{\frac{1}{2}} f_{3d}(Z_0, r_0),$$

where $r_0 = ((Z-\sigma)/(Z_0-\sigma))r$. Unfortunately the value of σ depends on the choise of the scale factor R, so that one cannot speak of a screening « constant ». Every atomic property to be calculated requires its own σ. For example different σ's must be used to calculate the parameters F_k and the mean values of r^{-3}. Our criterion for the choice of R is to take a length in the calculation of which is predominant that part of the wave-function which is important in the calculation of the parameters we are interested in. In the calculation of F_k we choose as R the value of r for which f_{3d} is a maximum, whereas in the calculation of $\langle r^{-3} \rangle$ we choose the value of r for which the probability of finding the particle is one half the maximum value. Let σ_M and σ_P be the values of σ in these two cases.

Values of σ in the case where R is the mean value of r, represented by $\bar{\sigma}$, were given by HARTREE ([2]) who based them on preliminary results for V^{2+}, Mn^{2+}, Zn^{2+}. This cannot be done in our case, since for other elements we do not possess self-

consistent calculations such as those of Worsley. However in order to deduce σ_M and σ_P we use Hartree's $\bar{\sigma}$. As can easily be shown

$$(2) \qquad \begin{cases} \sigma_M \overset{\text{def}}{=} 1.36\,\bar{\sigma} - 0.36\,Z\,, \\ \sigma_P = 1.57\,\bar{\sigma} - 0.57\,Z\,. \end{cases}$$

The first equation was given by Hartree ([2]), and also holds for the values of V^{2+} given by Worsley. The second equation was obtained from Worsley's table by inspection. Assuming these relations are valid for all Z, two groups of approximated wave-functions can be obtained from (1), the typical expression being

$$(3) \qquad f_{3d}(Z, \sigma, r) = r^3 A_\sigma \exp[-a_\sigma r] + B_\sigma \exp[-b_\sigma r] + C_\sigma \exp[-c_\sigma r])\,.$$

Using the group of functions with $\sigma = \sigma_M$ the values of F_k were calculated. As is well known ([6]) they are given in atomic units by

$$F_k = \frac{F^k}{D_k} = \frac{1}{D_k} \int_0^\infty \int_0^\infty \frac{r_<^k}{r_>^{k+1}} f_{3d}^2(r_1) f_{3d}^2(r_2)\, \mathrm{d}r_1\, \mathrm{d}r_2\,,$$

where $r_<$ and $r_>$ are the lesser and the greater of r_1 and r_2. If we disregard the first term of (3), whose contributions are of the order of $5 \cdot 10^{-4}$ for F_2 and $4 \cdot 10^{-5}$ for F_4 by putting

$$f_{3d}^2(r) = r^6 \sum_{\lambda=1}^{3} B_\lambda \exp[-b_\lambda r]\,,$$

we have

$$F_k = \frac{1}{D_k} \sum_{\lambda,\mu=1}^{3} B_\lambda B_\mu \frac{(5-k)!\,(6+k)!}{(b_\lambda + b_\mu)^{13}} \sum_{r=0}^{5-k} \binom{6+k+r}{r} \frac{b_\lambda^{k+r-6} + b_\mu^{k+r-6}}{(b_\lambda + b_\mu)^{k+r-6}}\,.$$

In the case of $3d$ electrons $k = 2.4$ and D_k is given on p. 179 of ref. ([6]). The experimental values were given by Orgel ([7]). For Ni^{2+} the values of F_2 and F_4 were deduced from the experimental values of energy levels given by Shenstone ([8]) using the relations between the energy levels and the Slater-Condon parameters for the $3d^2$ electronic configuration ([6]). In Table I are compared the experimental and calculated values.

Table I also shows the calculated values of $\langle r^{-3} \rangle$ usng the group of wave-functions with $\sigma = \sigma_P$. In the usual manner

$$\langle r^{-3} \rangle = \int_0^\infty f_{3d}^2(\sigma_P, r)\, \frac{\mathrm{d}r}{r^3}\,,$$

([6]) E. U. Condon and G. H. Shortley: *Theory of Atomic Spectra* (New York, N.Y., 1935).
([7]) L. E. Orgel: *Journ. Chem. Phys.*, **23**, 1819 (1955).
([8]) A. G. Shenstone: *Journ. Opt. Soc. Am.*, **44**, 749 (1954).

TABLE I. – *Experimental and theoretical values in a.u.*

Z	$\langle r^{-3} \rangle$ calc.	$\langle r^{-3} \rangle$ exp.	$10^2 F_2$ calc.	$10^2 F_2$ exp.	$10^2 F_4$ calc.	$10^2 F_4$ exp.
22	1.8	—	0.523	0.508	0.036	0.037
23	2.3	2.3	0.576	0.573	0.040	0.039
24	2.9	—	0.627	0.618	0.044	0.039
25	3.5	3.5	0.678	0.657	0.048	0.044
26	4.3	—	0.730	0.705	0.051	0.055
27	5.1	5.2	0.783	0.733	0.055	0.055
28	6.1	—	0.837	0.859	0.059	0.062
29	7.1	6.3	—	—	—	—

and the third term of equation (3) was neglected on account of its negligible contribution. The experimental values are those of ABRAGAM *et al.* (9).

The good agreement between the experimental and calculated results suggests that the two sets of analytical expressions can be taken as reasonable approximations of the $3d$ wave-functions for the doubly-ionized elements of the iron group. The functions with $\sigma = \sigma_P$ are an approximation for small values of r, and those with $\sigma = \sigma_M$ for large values of r. The values of σ cannot, however, be regarded as well established, and there is a slight error in the values of F_2 and F_4 calculated with the analytical expressions on account of the poor approximation of the values of Worsley, by this expression for large r. With regard to the calculation of $\langle r^{-3} \rangle$, since the experimental values refer to elements in combination, the above agreement must be considered as qualitative only.

* * *

We wish to thank Prof. F. G. FUMI and Prof. F. BASSANI for valuable discussions and Prof. M. SANTANGELO for his interest in the work. The thanks of one of us (C.C.) are due to the C.N.R. for a grant.

(9) A. ABRAGAM, J. HOROWITZ and M. H. L. PRYCE: *Proc. Roy. Soc*, **230**, 169 (1955).

IL NUOVO CIMENTO Vol. XIV, N. 3 1° Novembre 1959

On the Interactions of the Nucleons.

L. Tenaglia

Istituto di Fisica dell'Università - Bari

(ricevuto il 19 Agosto 1959)

In order to get an unified scheme for the equations of motion of the baryons, which may be very useful in investigating their symmetry properties, Gürsey [1] and Dallaporta and Toyoda [2,3] have recently studied generalized Dirac equations for 2.4 and 2.8-component spinors representing, respectively, the motion of two and eight fermions (while « 4 » stands for the number of components in the relativistic space). A problem which was never exactly nor satisfactorily treated by the above mentioned authors concerns the electromagnetic interaction of the baryons, although their procedure can give easily an answer to the problem in the case of $2 \cdot n$ spinors (n charged and n uncharged, as is the case for the eight known baryons). In this preliminary note we want to discuss very rapidly the Dirac equation for the nucleon field ($n = 1$) and to compare the electromagnetic interaction to the pion and Fermi interaction of the nucleons; indeed, one can give a unitary geometrical picture for the above mentioned interactions.

To deduce the Dirac equation for the nucleon field, we use the DT formalism, slightly modified to get a general form, which turns out very useful in connection with the usual isotopic spin formalism; namely, we define the DT's matrices Γ^μ, Γ_5 and the spinor X by means of the relations

$$\Gamma^\mu = \tau_1 \gamma^\mu , \qquad \Gamma_5 = -\tau_3 \gamma^5 , \qquad X = \begin{pmatrix} \chi \\ \overline{\chi} \end{pmatrix} , \tag{1}$$

where τ_1, τ_3 are the two real matrices of the set of three Pauli matrices τ_i: τ_i and γ^μ commute, obviously. The Gürsey equation

$$(\tau_1 \gamma^\mu \partial_\mu - im\tau_3 \gamma^5) X = 0 , \tag{2}$$

(1) G. Gürsey: *Nuovo Cimento*, **7**, 411 (1958).

(2) N. Dallaporta and T. Toyoda: *On the Dirac Equation or Barions* (preprint, in the following denoted by DT).

(3) N. Dallaporta and T. Toyoda: *On Transformation Properties of Strong Interactions* (preprint).

can be reduced by means of the unitary trasformation

$$X = TY\,, \qquad T^{-1} = T = \frac{1}{2}(\tau_1 + \tau_3 + (\tau_1 - \tau_3)\gamma^5)\,, \qquad Y = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{3}$$

to a system of two usual Dirac equations for the spinors ψ_1, ψ_2: the reducibility is guaranted by the commutation relation $(\tau_1\gamma^\mu, \tau_3\gamma^5) = 0$. To obtain the Dirac equation in the presence of an electromagnetic field, we take in eq. (1) the substitution

$$\gamma^\mu \partial_\mu \to \gamma^\mu \left(\partial_\mu - \frac{ie}{2}\tau_1 (1 - \tau_3\gamma_5) A_\mu\right); \tag{4}$$

going back to the equations of motion for the Y-spinor, one obtains the two Dirac equations

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi_1 + im\psi_1 = 0\,, \qquad (\gamma^\mu \partial_\mu + im)\tilde{\psi}_2 = 0\,. \tag{5}$$

Thus, one can get a representation of the Gürsey spinor in which the electromagnetic field, introduced by means of the substitution of eq. (4), acts on one component of Y only. An equation similar to the eq. (5) — but with $im \to im\gamma^5$ — may be obtained starting from another Gürsey equation ($\tau_3\gamma_5 \to \tau_3$) by the same procedure.

It is interesting to find out a geometric picture of the preceding result, and it may be the following. In a 3-dimensional space we fix two axes x_1 and x_2 and two planes orthogonal to these, (π_1 and π_2), and a plane containing them (π). Inversions [4] (including the identity transformation), accomplished by means of the Pauli matrices τ_i, give rise to an arbitrary rotation, in the π plane, of a vector: special cases are the rotation of a vector, brought to coincide with a vector along the x_1 axis (that is, in a particular frame of reference, the identity transformation) and the exchange amongst the x_1 and x_2 axes (special rotation by an angle $\vartheta = \pi/2$). The last transformation is a linear function of the τ_1, τ_2 matrices, the former is a linear function of τ_1, τ_3 and it fixes an axis (and therefore the whole x_1, x_2 frame of reference) in the π plane.

Going back to the physical problem, we take into account, beyond the electromagnetic, the pionic and the Fermi interactions amongst nucleons. The Fermi interaction is assumed to satisfy the $\Delta Q = +1$ condition, that is, it must involve spinors whose electrical charge differs by a value of modulus unity: for simplicity's sake, we suppose that such an interaction is established by a vector charged meson field $b^i_\mu \to b^\pm_\mu$; that is, the Fermi interaction hamiltonian is supposed to be (at the nucleon's decay vertex)

$$\mathscr{H}_F = \text{const}\ \tilde{\psi}\tau_i\gamma^\mu(1 - \gamma^5)\psi b^i_\mu\,. \tag{6}$$

It is obvious that the three fundamental interactions of the nucleon field are associated to, and pictured by, the above mentioned three kinds of transformations in the π plane of the x_1, x_2 axes: more precisely, the transformation of eq. (3) fixes the « proton axis » which we take as corresponding to the x_1 axis.

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(4) Such problems are treated, for instance, by H. BÖRNER: *Darstellungen von Gruppen* (Berlin, 1955), chapts. 8 and 9.

As to the electromagnetic interaction, it is well known that this can be deduced from the gauge invariance of first kind. The gauge transformation group is isomorph (5) to the one-dimensional rotations group; if we refer ourselves to the transformation $x_1 \to \psi_p$ (ψ_p proton spinor), such a rotation in the π_1 plane implies the mapping $-x_1 \to \psi_p^c$. Indeed, by introducing in equations (1)-(4) the charge conjugate operators, one gets the equations

$$(5') \qquad \gamma^\mu(\partial_\mu + ie A_\mu)\psi_1^c - im\psi_1^c = 0\,, \qquad (\gamma^\mu \partial_\mu - im)\psi_2^c = 0\,.$$

Similar relations can be obtained for the pion and Fermi interactions of the nucleon-antinucleon field introducing the well known 4×4 isotopic spin matrices.

Both with the DT procedure and with the equation mentioned below form. (5), it can be seen that the electromagnetic and Fermi interactions are not independent from rotations in isotopic space. One should therefore, in any case, fix by means of a unitary transformation, the charged eigenstate amongst the vector states.

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The Multiple Production of Particles in (pp)- and (pπ⁻)-Collisions at Energies of (1 ÷ 10) GeV.

V. S. Barašenkov

Joint Institute for Nuclear Research, Laboratory of Theoretical Physics - Dubna

(ricevuto l'11 Settembre 1959)

In Figs. 1 and 2 the calculated probabilities W_n of the 2-, 4-, 6-prong star production and the calculated number of the charged particles $\bar{n}$, produced in (π^-p) and (pp)-collisions are given. The interaction between the created π-mesons and the nucleons was taken into account [1].

All other assumptions and the method of the calculations are the same as in the papers [2]. When drawing the curves the results of the calculations [1,2] were used. The experimental

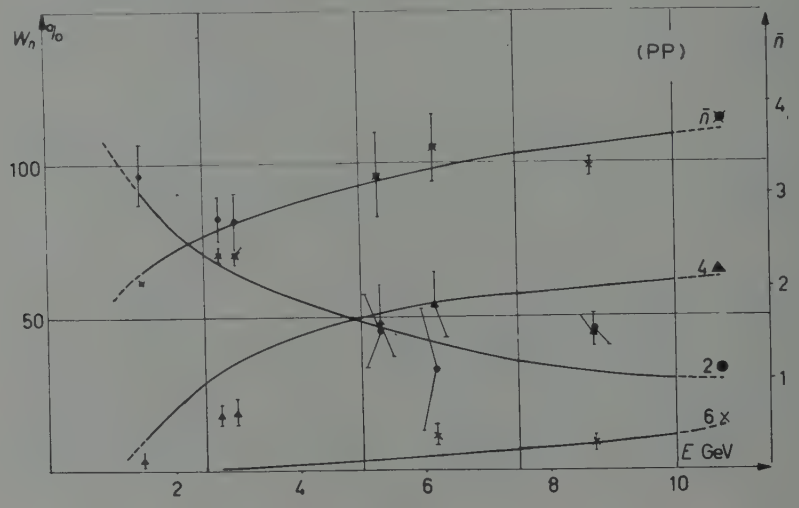

Fig. 1.

[1] S. Z. Belenky *et al.*: *Usp. Fis. Nauk*, **62**, n. 2 (1957).

[2] V. S. Barašenkov, B. M. Barbašev, E. G. Bubelev and M. V. Maksimenko: *Nucl. Phys.*, **5**, 17 (1957); **7**, 146 (1958); *Suppl. Nuovo Cimento*, **7**, 117 (1958); *Acta Phys. Pol.*, **17**, 177, 397 (1958).

data ([3-5]) are given for comparison. The statistical errors ΔW_n and the dispersion $\Delta\bar{n}$ are also indicated. Within these errors the experimental data are close to the theoretical ones. However, the experimental number of the two prong stars in (pp)-collisions is greater than the theoretical one. Yet, the number of the found events is insufficient to draw the conclusions about many-prong stars ($n > 4$).

The qualitative difference is observed if the theoretical angular prong distributions are compared with the experimental ones: the theoretical distributions are isotropic (in the c.m.s.), but the experimental ones are non-isotropic and even asymmetrical with respect to the angle $\theta = \pi/2$ (in (π^-p)- and (pn)-collisions ([4,5])).

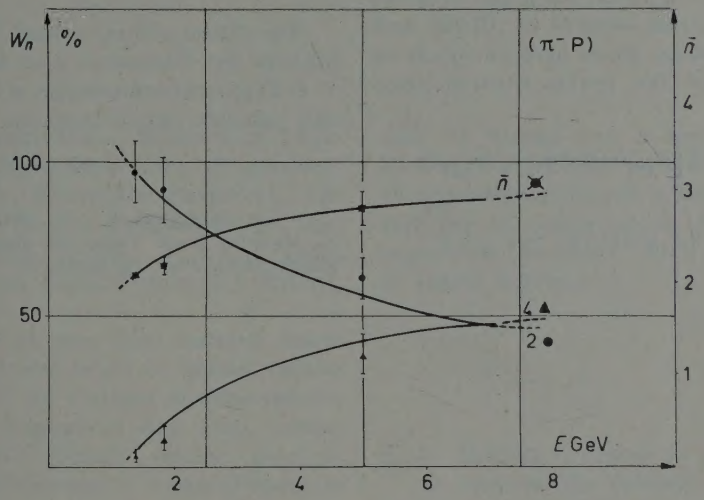

Fig. 2.

To account for the differences following ([6]) we take into consideration the peripheral collisions. It is possible to explain the asymmetry in (pn)-collisions, if the nucleon, which lost the virtual π-meson, passes into an excited state (isobar), which further decays (*). The statistical theory of multiple production was used and the diffraction scattering was taken into account in the calculations of the peripheral π-meson collisions with a nucleon and π-meson. It is possible

([3]) W. B. Fowler, R. W. Wright, G. Saphir, W. M. Powell and G. Maenchen: *Phys. Rev.*, **100**, 1802 (1955); W. B. Fowler, R. P. Shutt, A. M. Thorndike and W. L. Whittemore: *Phys. Rev.*, **103**, 1479 (1956); M. M. Block, E. M. Hart, V. T. Cocconi, E. Hart, W. B. Fowler, R. P. Shutt, A. M. Thorndike and W. L. Whittemore: *Phys. Rev.*, **103**, 1484 (1956); R. Cester, T. F. Hoang and A. Kernan: *Phys. Rev.*, **103**, 1443 (1956); R. M. Kalbach, J. J. Lord and C. H. Tsao: *Phys. Rev.*, **113**, 330 (1959); L. M. Eisberg, W. B. Fowler, R. M. Lea, W. D. Shephard, R. P. Shutt, A. M. Thorndike and W. L. Whittemore: *Phys. Rev.*, **97**, 797 (1955); R. C. Whitten and M. M. Block: *Phys. Rev.*, **111**, 1676 (1958); N. P. Bogachev, S. A. Buniatov, Iu. P. Merekov and V. M. Sidorov: *Dokl. Akad. Nauk USSR*, **121**, 617 (1958).

([4]) G. Maenchen, W. B. Fowler, W. M. Powell and R. W. Wright: *Phys. Rev.*, **108**, 850 (1957).

([5]) N. Bogachev *et al.*: *Journ. Exper. Theor. Phys.* (in press).

([6]) D. Blohintsev: *CERN Symposium*, **2**, 155 (1956).

(*) After the calculations have been made V. I. Veksler informed me, that I. E. Tamm had examined also the isobar in the peripheral collisions (unpublished).

to explain the experimental data, if

$$\xi \equiv \sigma^{p}_{\mathcal{N}\mathcal{N}}/\sigma_{\mathcal{N}\mathcal{N}} \geqslant 0.2 \div 0.3 ,$$

and

$$\eta \equiv \sigma^{p}_{\pi\mathcal{N}}/\sigma_{\pi\mathcal{N}} \geqslant 0.2 ;$$

where σ^{p} is the cross section of the peripheral collisions; $\sigma_{\mathcal{N}\mathcal{N}}$ and $\sigma_{\pi\mathcal{N}}$ are the total cross sections of $(\mathcal{N}\mathcal{N})$- and $(\pi\mathcal{N})$-collisions. From here we obtain an estimate of the $(\pi\pi)$-interaction cross section

$$\sigma^{p}_{\mathcal{N}\mathcal{N}} = 2\int \sigma_{\pi\mathcal{N}}(\varepsilon)\, q(\varepsilon)\, \mathrm{d}\varepsilon \simeq 2\sigma_{\pi\mathcal{N}} \cdot n ;$$

$$\sigma^{p}_{\pi\mathcal{N}} \simeq \int \sigma_{\pi\pi}(\varepsilon)\, q(\varepsilon)\, \mathrm{d}\varepsilon \simeq \sigma_{\pi\pi} \cdot n ;$$

where $q(\varepsilon)$ is a peripheral meson spectrum (7). Since $\sigma_{\pi\mathcal{N}} \simeq \sigma_{\mathcal{N}\mathcal{N}}$ then

$$\sigma_{\pi\pi} \simeq \sigma_{\pi\mathcal{N}} 2\eta/\xi \sim \sigma_{\pi\mathcal{N}} .$$

* * *

The author wishes to thank D. I. BLOHINTSEV for discussions and advice and V. I. VEKSLER for discussion of the results and valuable critical remarks.

(7) V. BARAŠENKOV, V. MALTSEV and E. MICHUL: *Nucl. Phys.* (in press); *Journ. Exper. Theor. Phys.* (in press).

LIBRI RICEVUTI E RECENSIONI

F. WOODBRIDGE – *Theoretical Physics* (*Thermodynamics, Electromagnetism, Waves and Particles*). Addison-Wesley Publishing Co. Inc. Reading (Mass., U.S.A.), 1958, pp. XIII+364. Prezzo 7,5 dollari.

Il lettore non deve lasciarsi trarre in inganno dal titolo di questo volume ritenendolo un trattato di fisica teorica, come ad esempio il ben noto volume dello Joos. Questo volume contiene invece l'esposizione in forma elementare delle teorie classiche concernenti le parti della fisica indicate nel sottotitolo. Questo tipo di trattazione risulta assai insolito allo studioso italiano abituato a trovarla unita, nei corsi di fisica sperimentale del primo biennio universitario, all'esposizione dei fatti sperimentali più importanti, che in questo volume l'autore ritiene già noti al lettore. Rilevata questa particolarità, dovuta all'ordinamento degli studi delle Università americane cui questo libro è dedicato, si può notare che l'esposizione della materia trattata è effettuata chiaramente ed è ricca di disegni esplicativi e di esempi ed esercizi. Anche i volumi consigliati al lettore per approfondire quanto viene trattato nei vari capitoli sono scelti con oculatezza.

Volendo però illustrare un considerevole numero di argomenti nello spazio di un solo volume l'autore non sempre riese a trattare con il dovuto approfondimento critico talune questioni fondamentali quali ad esempio la proprietà elettriche e magnetiche dei mezzi materiali. Il lettore non avvertito rischia spesso di non rendersi conto di quali siano gli argomenti principali e quali i secondari per la comprensione dei molti e importanti fenomeni fisici considerati in questo volume.

A. ALBERIGI

A. S. BISHOP – *Verso la produzione di energia da fusione* (*Il piano Sherwood*). Edizioni dell'Ateneo, Roma.

Si tratta della versione italiana, a cura del Prof. Ing. A. M. ANGELINI con la collaborazione del Dr. P. MARINI, di uno dei 12 volumi editi dalla Addison-Wesley e presentati dagli Stati Uniti nel Settembre 1958 alla seconda Conferenza Internazionale di Ginevra sugli usi pacifici dell'energia atomica. In esso l'autore A. S. BISHOP descrive con grande competenza lo sviluppo del programma americano Sherwood sulla fusione controllata dal 1951 al Settembre 1958.

L'esposizione generalmente segue il processo storico e rende conto dello sforzo organizzativo e delle principali linee seguite in America per affrontare il problema di portare un plasma a temperature astronomiche ed ottenere reazioni termonucleari controllate con elementi leggieri.

Nella descrizione delle macchine l'autore non trascura di accennare ai difficili problemi tecnologici e di fisica del plasma

incontrati e dà anche in giudizio sulle prospettive future delle varie linee di attacco finora seguite.

Era intenzione dell'autore di scrivere il libro anche per coloro che hanno scarsa o nessuna familiarità con l'argomento e addirittura non hanno un grande bagaglio di nozioni scientifiche.

Questa intenzione non è stata delusa, ma anzi magistralmente concretata in un'esposizione lineare e semplice, senza che ne scapiti il rigore di espressione; la comprensione è grandemente facilitata da un glossario di termini tecnici, utile per i non iniziati, e da numerosi disegni a colori di interpretazione immediata.

Queste caratteristiche si ritrovano fedelmente nell'edizione italiana: la traduzione è completa e ottima (a parte qualche rara espressione che può suonare ostica all'orecchio del fisico raffinato) e i disegni a colori sono identici a quelli dell'opera originale. Pertanto questa opera si raccomanda a tutti coloro che intendono informarsi ed eventualmente lavorare in questo importante nuovo settore; in particolare si raccomanda al fisico, all'ingegnere e al pianificatore di ricerca.

Avendo letto l'opera del BISHOP si ha un quadro completo di quanto è stato fatto circa la fusione nei laboratori americani e, grosso modo, in quelli del resto del mondo, perchè, come ha rivelato la conferenza di Ginevra dello scorso anno, i metodi impiegati sono circa gli stessi nei vari paesi.

B. BRUNELLI

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